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Abstract

A presentation is a triple $\langle X, \leq, M\rangle$ with $\langle X, \leq\rangle$ a finite poset and $M : X \to \mathcal{P}(\mathcal{P}(X))$ – these data being subject to additional constraints. Given a presentation we can define closed subsets of X, whence a finite lattice. Given a finite lattice L, we can define its presentation: X is the set of join-irreducible elements of L, $\leq$ is the restriction of the order to join-irreducible elements, and $M(x)$ is the set of minimal join-covers of x.

Morphisms of presentations can be defined as some zig-zag relations. Our main result is: *the category of presentations is dually equivalent to the category of finite lattice*. The two construction described above are the object part of contravariant functors giving rise to the duality.

We think of presentations as semantic domains for lattice terms and formulas of substructural logics. Relying on previous work by Nation and Semenova, we show that some equational properties of finite lattices correspond to first order properties of presentations. Namely, for each finite tree T, we construct an equation e_T that holds in a finite lattice if and only its presentation does not have some a shape from T. We illustrate further use of these semantics within the theory of fixed points over finite lattices: we generalize, in a non-trivial way, the well known fact that least fixed points on distributive lattice terms can be eliminated.

1 Introduction

We develop in this paper a duality theory for finite lattices conceived around the notion of the OD-graph of a finite lattice [22].

Many representation theorems are available for finite lattices [8] and, more generally, for lattices [30]. For finite lattices, the most known of these theorem represents a lattice as a concept lattice, built up from a table of objects and concepts that give rise to a Galois connection [11]. The scientific community has devoted much work to lifting this representation theorem to some full duality of categories [18, 17, 12].

We develop here a duality theory that we conceive from a relational structure – the OD-graph – that we can associate to a finite lattice, and on a standard representation theorem for finite lattices associated to such structure [22, Lemma 1.2]. The relational structure and the representation theorem are well known tools in lattice theory [10, 15, 28]; these are also well known among lattice theorists whose research is motivated from the theory of databases [4].

In our view, the duality theory that we develop possesses many points of interest. For example, it smoothly generalizes Stone-Priestly duality for finite distributive lattices. Also, we provide many examples of equational properties of lattices that can be translated into first order properties of their OD-graphs. While it is not the goal of this paper to develop a Sahlqvist-style correspondence theory, the examples that we exhibit are meant to suggest that the correspondence theory might be developed in more effective way for this duality than for other dualities.

The duality theory so developed shares many similarities with current research on duality for lattices based on the notion of closure space [23]. Also, the semantics it gives rise is in the same spirit of the covering semantics of linear logic proposed in [13]. Let us recall that the latter semantics was conceived from the topos theoretic sheaf semantics of intuitionistic higher order logic [7, 20]. These similarities are not, in our opinion, a fruit of hazard: they arise from the recognition – by many researchers from different perspectives and backgrounds – of the central role of some mathematical tools, such as the role of join-generation in lattices as well as the usefulness of a working duality theory as known from algebraic modal logic.

We begin our paper by recalling what it means for a lattice to have the minimal join-cover refinement property. Indeed, it is this property – that trivially holds in every finite lattice – that is at the heart of the duality. We have taken the path of developing the duality only for the finite case thus to focus on the combinatorial aspects of the duality. We decided to delay to further researches the problem of generalizing the duality to larger classes of lattices.

We introduce then the notion of join-presentation. By carefully defining the notion of arrow between join-presentations – that we call oriented bisimulation – we construct a full and faithful contravariant functor into the category of lattices. In a second step, we observe that the OD-graph of a lattice, a combinatorial structure introduced in [22] that encodes the essence of the minimal join-cover refinement property, gives rise to an adjoint functor in the opposite direction, from the category of lattices to the category of join-presentations and oriented bisimulations.

The notion of oriented bisimulation is quite involved, and time will decide about its usefulness. Yet, in case an oriented bisimulation corresponds under the duality to a lattice epimorphism, such a bisimulation can be easily described: each monic oriented bisimulation is witnessed by an embedding which is a zigzag morphism of all the relational structures that are part of a join-presentation. We illustrate the use of this restricted duality by computing the OD-graph of the Associahedra from the OD-graph of Pemutohedra.

Finally, we investigate what it means for a presentation to be, abstractly, an OD-graph of some lattice. We exhibit five axioms that enforce a presentation to be isomorphic to the OD-graph of some lattice. We develop further considerations in order to construct OD-graphs of atomistic lattice.

A second part of the paper is devoted to illustrate possible applications and developments of this duality. We exhibit an infinite sequence $\mathcal{D}_n$ of lattice varieties such that, on finite lattices and for any lattice polynomial ϕ, the approximants $\phi^i(\perp)$ converge to a fixed point after n steps, but that might take at least $\frac{n-2}{3}$ steps to converge.

Finally, we address the fact that many equational properties of a lattices can be translated into first order property of their OD-graphs. Our observation calls for a problem: what are the lattice equations that corresponds to first-order definable properties – or, more generally, reasonable properties – of an OD-graph? We invite the researchers to tackle answer this question, which has a strong analogy with what is called in modal logic correspondence theory.

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2 Notation, definitions, elementary facts

If $R \subseteq X \times Y$ is a relation, then we shall use the notation xR for the image of x under R, $xR = \{ y \in Y \mid xRy \}$.

For elementary notions about ordered sets and lattices, we invite the reader to consult the standard literature [14, 8]. Similarly, we address the reader to [21] for elementary facts on category theory. All the lattices we shall encounter shall be finite, unless explicitly stated.

If P is a poset and $p \in P$, then $\downarrow p$ shall denote the *principal ideal of* p, that is, the set $\{ p' \in P \mid p' \leq p \}$. Similarly, if $C \subseteq P$, then $\downarrow C$ denotes *the lower set generated by* C, $\downarrow C = \{ p \in P \mid \exists c \in C \text{ s.t. } p \leq c \}$.

If P, Q are posets and $f : P \to Q$ is an order preserving map, then we say that f is a *left adjoint* (or that it is *residuated*) if there exists a function $g : Q \to P$ such that $f(p) \leq q$ if and only if $p \leq g(q)$, for each $p \in P$ and $q \in Q$. Notice that g is uniquely determined by f, and f is uniquely determined by g, so that we say that g is *right adjoint* to f, and that f is left adjoint to g; we write then $f \dashv g$. In lattice theory, a lattice morphism that has a left adjoint is called a *lower bounded morphism*, and a morphism of lattices that has a right adjoint is called an *upper bounded morphism*.[1]

Let us recall the following fact: if L, M are complete lattice – in particular if they are finite – an order preserving map $f : L \to M$ is a left adjoint if and only if it preserves arbitrary joins. Similarly, f is a right adjoint if and only f preserves arbitrary meets. Thus, if L, M are finite, f is lattice morphism if and only if it is both a left adjoint and a right adjoint.

If L is a lattice, then $J(L)$ shall denote the collection of join-irreducible elements of L. If $x \in L$ and $Y \subseteq L$ is such that $x \leq \bigvee Y$, then we say that Y is a *join-cover* of x. We say that Y is a *non-trivial join-cover* of x if it is a join-cover of x and, moreover, $x \not\leq y$ for each $y \in Y$.

2.1 The minimal join-cover refinement property (MJCRP)

As the minimal join-cover refinement property has been the starting point for our investigation of duality for lattices, we previously recall what it means for a (possibly infinite) lattice to enjoy this property.

If S is a set, then $\mathcal{P}_{\mathrm{f}}(S)$ denotes the set of finite subsets of S. If P is a poset, then $\mathcal{P}_{\mathrm{f}}(P)$ is a preordered set, where the preorder is given by the refinement relation:

$$X \ll Y \quad \text{iff} \quad \forall x \in X \exists y \in Y \text{ s.t. } x \leq y \, .$$

For the moment, let us take the following as definition of the minimal join-cover refinement property:

[1] The two notions, that of lower bounded lattice morphism and that of a lattice morphism that has a left adjoint coincide if if the domain lattice has both a least and a greatest elements.

Definition 2.1. A lattice L has the *minimal join-cover refinement property* if there exists a function $\mathcal{M} : L \to \mathcal{P}_{\mathfrak{f}}(\mathcal{P}_{\mathfrak{f}}(L))$ such that, for all $x \in L$,

$$x \leq \bigvee Y \quad \text{iff} \quad \exists C \in \mathcal{M}(x) \, \text{s.t.} \, C \ll Y \, . \tag{1}$$

Our first remark is that the function $\mathcal{M}$ – whenever it exists – is not uniquely determined. Yet, if a function $\mathcal{M}$ satisfying (1) exists, then there is another function satisfying (1) and such that:

$$\mathcal{M}(x) \text{ is an antichain w.r.t. } \ll \, , \tag{2}$$

$$C \text{ is an antichain w.r.t. } \leq \, , \tag{3}$$

for each $x \in L$ and $C \in \mathcal{M}(x)$. As a matter of fact, if $\mathcal{M}(x)$ is not an antichain then we can replace it with the set of its minimal elements. If some $C \in \mathcal{M}(x)$ is not an antichain, then we can replace it with the set of its maximal elements.

Lemma 2.2. *There exists at most one function $\mathcal{M}$ satisfying* (1), (2), *and* (3). *For such a function, the following holds:*

1. *if $c \in C \in \mathcal{M}(x)$, then $c \in J(L)$,*
2. *if $j \in J(L)$, then $\{ j \} \in \mathcal{M}(j)$,*
3. *if x is a join-reducible element, then we can write it as $x = j_1 \vee \ldots \vee j_n$, $n > 1$, where $j_i \in J(L)$ for $i = 1, \ldots, n$. Moreover, if $D \in \mathcal{M}(x)$ then D is the set maximal elements of a set of the form $\bigcup_{i=1,\ldots,n} C_i$, where $C_i \in \mathcal{M}(j_i)$ for $i = 1, \ldots, n$.*

Proof. We only prove the three properties.

Let $c \in C \in \mathcal{M}(x)$. If $c = y \vee z$ and $c \in C$, then $D = C \setminus \{ y, z \}$ is such that $D \ll C$. Moreover D is a cover of x, whence there exists $E \in \mathcal{M}(x)$ such that $E \ll D$. As $\mathcal{M}(x)$ is an antichain, then $E = C$, and C is the set of maximal elements of D. In particular $c \in D$ and since C is an antichain, either $c = y$ or $c = z$.

Observe that $j \leq \bigvee \{ j \}$ so that we can find $C \in \mathcal{M}(j)$ such that $C \ll \{ j \}$. Clearly, $j = \bigvee C$, whence $j \in C$. Since C is an antichain, if $k \in C$ with $k \neq j$, then $j < \bigvee C$. Whence we have $C = \{ j \}$.

Finally, let us write $x = y \vee z$ with $x \neq y$ and $x \neq z$. Then there exists $C \in \mathcal{M}(x)$ such that $C \ll \{ y, z \}$ and $x = \bigvee C$. Every element of C is an element of $J(L)$, by *1* and C is not a singleton, since then $C = \{ x \}$ and $x \leq y$ or $x \leq z$. If $D \in \mathcal{M}(x)$, then $c \leq \bigvee D$, for each $c \in C$, so that there exists $E_c \in \mathcal{M}(c)$ such that $E_c \ll D$. It follows that $\bigcup_{c \in C} E_c \ll D$, and since $\bigcup_{c \in C} E_c$ is a cover of x, then $D \ll \bigcup_{c \in C} E_c \ll D$, by minimality. It follows that D is the set of maximal elements of $\bigcup_{c \in C} E_c$. ☐

The Lemma emphasizes that the fact that L is join-generated by its join-irreducible elements, and that $\mathcal{M}$ is determined by its restriction to join-irreducible elements. In the following we shall assume that $\mathcal{M}$ satisfies (1), (2), and (3).

Next, we want to compare our Definition 2.1 of the minimal join-cover refinement property to its usual definition, see for example [10, page 30]. For $j \in J(L)$, let us define

$$\mathcal{M}^{\mathrm{nt}}(j) = \mathcal{M}(j) \setminus \{ \{ j \} \} .$$

Lemma 2.3. *The collection $\mathcal{M}^{\mathrm{nt}}(j)$ is the set of minimal* non-trivial *join-covers of $j \in J(L)$, that is:*

1. *every $C \in \mathcal{M}^{\mathrm{nt}}(j)$ is a non-trivial join-cover of j,*
2. *every non trivial join-cover of j refines to a join-cover in $\mathcal{M}^{\mathrm{nt}}(j)$,*
3. *if $C \in \mathcal{M}^{\mathrm{nt}}(j)$ and Y is a join-cover of j with $Y \ll C$, then $C \subseteq Y$.*

Proof. If $C \in \mathcal{M}^{\mathrm{nt}}(j)$ then C is a join-cover of j: we deduce $j \leq \bigvee C$ from $C \in \mathcal{M}^{\mathrm{nt}}(j) \subseteq \mathcal{M}(j)$ and $C \ll C$. Moreover C is non-trivial: if $j \leq c$ for some $c \in C$, then $\{ j \} \ll C$, thus $C = \{ j \}$ as $\mathcal{M}(j)$ is an antichain. This however contradicts $C \in \mathcal{M}^{\mathrm{nt}}(j)$.

Let Y be a non-trivial join-cover of j. From $j \leq \bigvee Y$ we deduce $C \ll Y$ for some $C \in \mathcal{M}(j)$. If $C = \{ j \}$ then Y is trivial; whence $C \in \mathcal{M}^{\mathrm{nt}}(j)$.

Also, if $C \in \mathcal{M}(j)$, Y is a join-cover of j, and $Y \ll C$, then there exists a cover $C' \in \mathcal{M}(j)$ such that $C' \ll Y$. Thus $C' \ll C$, and considering that $\mathcal{M}(j)$ is an antichain, we have $C' = C$. As $C \ll Y \ll C$ and C is an antichain, we deduce $C \subseteq Y$. $\square$

To end this section, let us mention why we explicitly added $\{ j \}$ to $\mathcal{M}^{\mathrm{nt}}(j)$ to obtain $\mathcal{M}(j)$, and consider minimal join-covers of j instead of minimal non-trivial join-covers of j. Such a step is like adding a unit to binary associative operation, which eventually makes most of the proofs much smoother.

2.2 MJCRP : a conceptual understanding

Equation (1) – that on purpose we have emphasized by making it into the definition of the minimal join-cover refinement property – clearly exhibits the property as some kind of adjointness relation. We give next a detailed account of this fact, for which we make use of ideas from [9, 29] that we already exploited in [27].

Definition 2.4. Let P, Q preordered sets and let $f : P \to Q$ be an order-preserving map. We say that f is a *left $\mathcal{O}_{\mathfrak{f}}$-adjoint* if there exists $G : Q \to \mathcal{P}_{\mathfrak{f}}(P)$ such that, for any $p \in P$ and $q \in Q$, $f(p) \leq q$ if and only if $p \leq c$ for some $c \in G(q)$.

Let $\mathfrak{Pos}$ denote the category of posets (preordered sets) and order-preserving maps. Let also $\mathcal{O}_{\mathfrak{f}} : \mathfrak{Pos} \to \mathfrak{Pos}$ be the covariant functor that assigns to a poset the collection of finitely generated lower sets of P, ordered by inclusion. If $f : P \to Q$ is an arrow of $\mathfrak{Pos}$, then $\mathcal{O}_{\mathfrak{f}}(f)$ is defined by the following formula:

$$\mathcal{O}_{\mathfrak{f}}(f)(\bigcup_{i=1,\ldots,n} \downarrow p_j) = \downarrow f(\bigcup_{i=1,\ldots,n} \downarrow p_j) = \bigcup_{i=1,\ldots,n} \downarrow f(p_j)\,.$$

The following Lemma explains the naming.

Lemma 2.5. *An order preserving map $f : P \to Q$ is a left $\mathcal{O}_{\mathfrak{f}}$-adjoint if and only if $\mathcal{O}_{\mathfrak{f}}(f) : \mathcal{O}_{\mathfrak{f}}(P) \to \mathcal{O}_{\mathfrak{f}}(Q)$ is a left adjoint.*

Proof. Let us suppose that $\mathcal{O}_{\mathfrak{f}}(f)$ is such a left adjoint, with right adjoint H. Let $G(q)$ be the finite set of maximal elements of $H(\downarrow q)$. Then we have

$$\begin{aligned} f(p) \leq q \;\; & \text{iff} \;\; \mathcal{O}_{\mathfrak{f}}(f)(\downarrow p) \leq \downarrow q \\ & \text{iff} \;\; \downarrow p \leq H(\downarrow q) \\ & \text{iff} \;\; p \in H(\downarrow q) \\ & \text{iff} \;\; p \leq c, \text{ for some } c \in G(q)\,. \end{aligned}$$

Viceversa, let us assume that such a G exists, and define

$$H(\bigcup_{i=1,\ldots,n} \downarrow q_i) = \bigcup_{i=1,\ldots,n} \downarrow G(q_i)\,.$$

Since

$$\mathcal{O}_{\mathfrak{f}}(f)(\bigcup_{i=1,\ldots,n} \downarrow p_i) = \bigcup_{i=1,\ldots,n} \mathcal{O}_{\mathfrak{f}}(f)(\downarrow p_i)$$

and

$$\mathcal{O}_{\mathfrak{f}}(f)(\downarrow p) = \downarrow f(p)\,,$$

it is enough to verify that $f(p) \in \bigcup_{i=1,\ldots,n} \downarrow q_i$ if and only if $p \in \bigcup_{i=1,\ldots,n} \downarrow G(q_i)$, i.e. there exists $i \in \{1,\ldots,n\}$ and $c \in G(q_i)$ such that $p \leq c$. This immediately follows from the definition of a left $\mathcal{O}_{\mathfrak{f}}$-adjoint. $\square$

From now on we shall use $\mathcal{O}_{\mathfrak{f}}$-*adjoint* as a synonym of left $\mathcal{O}_{\mathfrak{f}}$-adjoint. We examine next the dual property of the one that defines $\mathcal{O}_{\mathfrak{f}}$-adjoint, that is:

there exists a function $G : Q \to P$ such that

$$q \leq f(p) \text{ if and only if } c \leq p \text{ for some } c \in G(q)\,.$$

The property states that $\mathcal{O}_{\mathfrak{f}}(f^{op}) : \mathcal{O}_{\mathfrak{f}}(P^{op}) \to \mathcal{O}_{\mathfrak{f}}(Q^{op})$ is a left adjoint, and therefore that $\mathcal{F}_{\mathfrak{f}}(f) : \mathcal{F}_{\mathfrak{f}}(P) \to \mathcal{F}_{\mathfrak{f}}(Q)$ is a right adjoint, where $\mathcal{F}_{\mathfrak{f}}$ is the functor assigning to a poset the poset of its finitely generated upper sets, ordered by reverse inclusion.

Definition 2.6. We say that f is a *dual* $\mathcal{O}_{\mathfrak{f}}$*-adjoint* if $\mathcal{O}_{\mathfrak{f}}(f^{op}) : \mathcal{O}_{\mathfrak{f}}(P^{op}) \to \mathcal{O}_{\mathfrak{f}}(Q^{op})$ is a left adjoint.

Remark 2.7. It is not correct to call a dual $\mathcal{O}_{\mathfrak{f}}$-adjoint "right $\mathcal{O}_{\mathfrak{f}}$-adjoint". As a matter of fact, $\mathcal{O}_{\mathfrak{f}}(f) : \mathcal{O}_{\mathfrak{f}}(P) \to \mathcal{O}_{\mathfrak{f}}(Q)$ is a right adjoint if and only if the following property holds: *there exists a function* $G : Q \to \mathcal{P}_{\mathfrak{f}}(P)$ *such that* $q \leq f(p)$ *if and only if* $c \leq p$ *for each* $c \in G(q)$.

Recall that the join function – which takes as argument a finite set – is an order preserving function of the following type:

$$\bigvee : \mathcal{P}_{\mathfrak{f}}(L) \to L \,,$$

where we recall that $\mathcal{P}_{\mathfrak{f}}(L)$ is ordered by the refinement relation. The discussion just presented then shows the following result:

Proposition 2.8. *A lattice has the minimal join-cover refinement property if and only if* $\bigvee$ *is a dual* $\mathcal{O}_{\mathfrak{f}}$*-adjoint.*

3 From presentations to lattices

We start here our investigation of the duality. As we have emphasized in the previous section, the minimal join-cover refinement property exhibits a lattice as join-generated from its join-irreducible elements. This lead us to focus on how finite lattices – that always are join-generated from their join-irreducible elements – compare to their join-semilattices reducts, and to the algebraic presentations of their reducts.

In principle, the goal of this section is to define a category of join-presentation and define a full and faithful contravariant functor to the category of lattices. In practice, we work with a different approach: we define a category of presentation by extending a map on objects and enforcing it a full and faithful contravariant functor, by looking for a clever definition of arrows between presentations.

Definition 3.1. A *join-presentation* is a triple of the form $\langle X, \leq, M\rangle$ where $\langle X, \leq\rangle$ is a finite poset and $M : X \to P(P(X))$.

In the following we shall use the word presentation as a synonym of join-presentation.

Recall that a downset of $\langle X, \leq\rangle$ is a subset $S \subseteq X$ such that $y \leq x \in S$ implies $y \in S$.

Definition 3.2. Let $\langle X, \leq, M\rangle$ be a presentation. A downset S of $\langle X, \leq\rangle$ is said to be *closed* if $C \in M(x)$ and $C \subseteq S$ imply $x \in S$. We let $\mathfrak{L}(X, \leq, M)$ be the poset whose elements are closed downsets of $\langle X, \leq\rangle$ and whose ordering is subset inclusion.

As usual $\mathfrak{L}(X, \leq, M)$ is a lattice, as an arbitrary intersection of closed downsets is a closed downset.

Let us denote by $\mathcal{O}(X, \leq)$ the set of downsets of a finite poset $\langle X, \leq \rangle$. Let us recall that $\mathcal{O}(X, \leq)$ is the *free join-semilattice* over the poset $\langle X, \leq \rangle$. This means, firstly, that $\mathcal{O}(X, \leq)$ is a join-semilattice, with union as the join operation and inclusion as the order. Secondly, the map that associates to $x \in X$ its principal ideal $\downarrow x \in \mathcal{O}(X, \leq)$ is an order embedding $\downarrow : \langle X, \leq \rangle \to \mathcal{O}(X, \leq)$ with the following universal property: if $f : \langle X, \leq \rangle \to L$ is an order preserving map and L is a join-semilattice, then there exists a unique join-preserving map $\tilde{f} : \mathcal{O}(X, \leq) \to L$ such that $\tilde{f}(\downarrow x) = f(x)$, for each $x \in X$.

The next Lemma gives an algebraic characterization of the join-semilattice $\mathfrak{L}(X, \leq, M)$.

Lemma 3.3. *The closure operation* $\overline{(\,\cdot\,)} : \mathcal{O}(X, \leq) \to \mathfrak{L}(X, \leq, M\rangle$ *is a join-semilattice epimorhism from* $\mathcal{O}(X, \leq)$ *to* $\mathfrak{L}(X, \leq, M)$ *whose kernel is the least congruence (of join-semilattices) enforcing the relations*

$$\downarrow x \leq \bigvee_{c \in C} \downarrow c\,, \quad \textit{for } C \in M(x)\,.$$

By saying that a congruence relation enforces a relation $\downarrow x \leq \bigvee_{c \in C} \downarrow c$ we simply mean that it contains the pair $(\downarrow x \cup \downarrow C, \downarrow C)$. Such a characterization can be found for example in [19]. We add a proof here, as it will turn out to be useful later.

Proof. That the closure operation $\overline{(\,\cdot\,)}$ is a join-semilattice epimorhism follows from the fact that the joins are computed in $\mathfrak{L}(X, \leq, M)$ as closure of unions, so that

$$\overline{\bigcup_{j \in J} I_j} = \overline{\bigcup_{j \in J} \overline{I_j}} = \bigvee_{j \in J} \overline{I_j}\,.$$

It is easily seen that $x \in \overline{\downarrow C}$ when $C \in M(x)$, so that

$$\overline{\downarrow x} \subseteq \overline{\downarrow C} = \bigvee_{c \in C} \overline{\downarrow c}\,.$$

Let us suppose, next, that $f : \mathcal{O}(X, \leq) \to L$ is a join-homomorphism such that $f(\downarrow x) \leq \bigvee_{c \in C} f(\downarrow c)$ whenever $C \in M(x)$; we shall show that $f(S) = f(\overline{S})$.[2] This in turn shall imply that $f : \mathfrak{L}(X, \leq, M) \to L$ is a join-homomorphism:

$$f(\bigvee_{j \in J} I_j) = f(\overline{\bigcup_{j \in J} I_j}) = f(\bigcup_{j \in J} I_j) = \bigvee_{j \in J} f(I_j)\,.$$

[2]Using the standard representation of quotients by equivalence classes, the fact that $f(S) = f(\overline{S})$ amounts to saying that f is well defined on equivalence classes.

Clearly we have $f(S) \leq f(\overline{S})$, as $S \subseteq \overline{S}$. In the other direction, let us define

$$F(Y) = \downarrow\{\, z \mid \exists C \in M(z) \,\text{s.t.}\, C \subseteq Y \,\}\,, \qquad G(Y) = S \cup F(Y)\,,$$

so that

$$\overline{S} = \bigcup_{n \geq 0} G^n(\emptyset)\,.$$

In order to prove that $f(\overline{S}) \leq f(S)$ it shall be enough to show that $f(Y) \leq f(S)$ implies $f(G(Y)) \leq f(S)$. We have

$$\begin{aligned} f(F(Y)) = f(\bigcup_{C \subseteq Y, C \in M(z)} \downarrow z) &= \bigvee_{C \subseteq Y, C \in M(z)} f(\downarrow z) \\ &\leq \bigvee_{C \subseteq Y, C \in M(z)} f(\downarrow C) \leq f(Y) \leq f(S)\,, \end{aligned}$$

whence

$$f(G(Y)) = f(S \cup F(Y)) = f(S) \vee f(F(Y)) \leq f(S)\,. \qquad \square$$

3.1 Simulations between presentations

We restrict next our attention to direct presentations, where we use the naming of [4]. A presentation $\langle X, \leq, M \rangle$ is *direct* if

$$\overline{S} = \{\, y \mid \exists C \in M(y) \ \text{s.t.}\ C \subseteq S \,\}\,, \tag{4}$$

for each $S \in \mathcal{O}(X, \leq)$. It is certainly possible to develop a more general theory, considering presentations that do not have this property. The result, however, might be a less elegant theory.

Lemma 3.4. *A presentation is direct if and only if the following conditions hold:*

1. *$y \leq x$ and $C \in M(x)$ implies $D \ll C$ for some $D \in M(y)$,*
2. *there exists $C \in M(x)$ with $C \ll \{\, x \,\}$,*
3. *if $C \in M(x)$ and $D_c \in M(c)$, for $c \in C$, then $E \ll \bigcup_{c \in C} D_c$, for some $E \in M(x)$.*

Proof. Let $T(S)$ be the expression on the right of (4), so that a presentation $\langle X, \leq, M \rangle$ is direct if and only if T is a closure operator of the lattice $\mathcal{O}(X, \leq)$.

Let us see first that T sends downsets to downsets iff condition (1) holds. Let us suppose that $T(S)$ is a downset whenever S is a downset; let moreover $C \in M(x)$ and $y \leq x$. Then $x \in T(\downarrow C)$ thus $y \in T(\downarrow C)$, as $T(\downarrow C)$ is a downset. By definition of T, there exists $D \in M(y)$ with $D \subseteq \downarrow C$, that is, $D \ll C$. Conversely, let assume that (1) holds, let S be a downset. Let

$y \leq x \in T(S)$: there exists $C \in M(x)$ with $C \subseteq S$. By (1), there exists $D \in M(y)$ with $D \ll C$, thus $D \subseteq S$. It follows that $y \in T(S)$.

Let us see next that $S \subseteq T(S)$, for each downset S, iff condition (2) holds. If $\downarrow x \subseteq T(\downarrow x)$, then $x \in T(\downarrow x)$, thus therte exists $C \in M(x)$ with $C \subseteq \downarrow x$, i.e. $C \ll \{ x \}$. Conversely, if $x \in S$ and $C \in M(x)$ is such that $C \ll \{ x \}$, then $C \subseteq S$, therefore $x \in T(S)$.

Finally, let us show that $T(T(S)) \subseteq T(S)$, for each downset S, iff condition (3) holds. Let us assume that $T(T(S)) \subseteq T(S)$, for each downset S. let $C \in M(x)$ and, for each $c \in C$, let $D_c \in M(c)$. Thus we have $x \in TT(\bigcup_{c \in C} \downarrow D_c) \subseteq T(\bigcup_{c \in C} \downarrow D_c)$ and there exists $E \in M(x)$ with $E \subseteq \bigcup_{c \in C} \downarrow D_c$, i.e. $E \ll \bigcup_{c \in C} D_c$. Conversely, let us assume that condition (3) holds, and let $x \in TT(S)$. There exists $C \in M(x)$ with $C \subseteq T(S)$. The last condition is equivalent to saying that, for each $c \in C$, there exists $D_c \in M(c)$ such that $D_c \subseteq S$. Let $E \in M(x)$ with $E \ll \bigcup_{c \in C} D_c$: then $E \subseteq S$, hence $x \in T(S)$. $\square$

Let us recall that a *bimodule* from $\langle X, \leq \rangle$ to $\langle Y, \leq \rangle$ is a relation $R \subseteq X \times Y$ which is a downset of $X^{op} \times Y$, i.e. such that $p' \geq pRq \geq q'$ implies $p'Rq'$. Observe that these bimodules are in a bijective correspondence with order preserving maps from $\langle X, \leq \rangle$ to $\mathcal{O}(Y, \leq)$:

$$y \in f_R(x) \quad \text{iff} \quad xRy \, .$$

Thus, by the freeness property of $\mathcal{O}(X, \leq)$, we immediately have

Lemma 3.5. *There is a bijection between bimodules from $\langle X, \leq \rangle$ to $\langle Y, \leq \rangle$ and join-homomorphisms from $\mathcal{O}(X, \leq)$ to $\mathcal{O}(Y, \leq)$.*

The inverse bijections are as follows: given R, F_R is defined by

$$F_R(I) = \bigcup_{i \in I} iR \, ,$$

and, given F, R_F is defined as

$$xR_Fy \quad \text{iff} \quad y \in F(\downarrow x) \, .$$

Definition 3.6. A bimodule R from $\langle X, \leq \rangle$ to $\langle Y, \leq, M \rangle$[3] is said to be *closed* if $xR = \{ y \mid xRy \}$ is a closed set, for each $x \in X$.

Let us remark that the above condition that defines closed bimodules is the same as saying that the sets

$$F_R(\downarrow x) = f_R(x)$$

are closed. Again, by freeness of $\mathcal{O}(X, \leq)$, we obtain

Lemma 3.7. *There is a bijection bewteen closed bimodules from $\langle X, \leq \rangle$ to $\langle Y, \leq, M \rangle$ and join-homomorphisms from $\mathcal{O}(X, \leq)$ to $\mathfrak{L}(Y, \leq, M)$.*

[3] We need M as part of $\langle Y, \leq, M \rangle$ in order to give a meaning to closedness.

In this case the inverse bijections are as follows: given R, F_R is defined by

$$F_R(I) = \overline{\bigcup_{x \in I} xR}\,,$$

where R_F, given F, is defined as before: xR_Fy iff $y \in F(\downarrow x)$.

Definition 3.8. A closed bimodule R from $\langle X, \leq \rangle$ to $\langle Y, \leq, M \rangle$ is said to be a *simulation* if

$$xRy \text{ and } C \in M(x) \text{ implies } D \subseteq \bigcup_{c \in C} cR, \text{ for some } D \in M(y)\,, \tag{5}$$

for each $x \in X$, $C \in M(x)$, and $y \in Y$.

Lemma 3.9. *For a closed bimudule R from $\langle X, \leq \rangle$ to $\langle Y, \leq, M \rangle$, the following conditions are equivalent:*

1. *R is a simulation,*
2. *$F_R(\downarrow x) = F_R(\overline{\downarrow x})$ and $F_R : \mathcal{O}(X, \leq) \to \mathfrak{L}(Y, \leq, M)$ restricts to a join-homomorphism $\mathfrak{L}(X, \leq, M) \to \mathfrak{L}(Y, \leq, M)$,*
3. *$F(\overline{S}) = F(S)$, for each $S \in \mathcal{O}(X, \leq)$.*

Proof. (2) implies (1): let us assume that the restriction of F_R to $\mathcal{L}(X, \leq, M)$ is a join-homomorphism and verify that (5) holds. If $C \in M(x)$, then $x \in \overline{\downarrow C}$ and

$$\begin{aligned} y \in F_R(\downarrow x) &\subseteq F_R(\overline{\downarrow C}) \\ &= F_R(\bigvee_{c \in C} \overline{\downarrow c}) = \bigvee_{c \in C} F_R(\overline{\downarrow c}) \\ &= \bigvee_{c \in C} F_R(\downarrow c) = \overline{\bigcup_{c \in C} F_R(\downarrow c)}\,. \end{aligned}$$

As $\langle Y, \leq, M \rangle$ is direct, we deduce that there exists $D \in M(y)$ with $D \subseteq \bigcup_{c \in C} F_R(\downarrow c)$.

(1) if and only if (3): we have already seen in Lemma 3.3, that $F_R(S) = F_R(\overline{S})$, if and only if $F_R(\downarrow x) \subseteq F_R(\downarrow C)$ whenever $C \in M(x)$. The last condition is equivalent to xRy and $C \in M(x)$ implies $y \in F_R(\downarrow C) = \overline{\bigcup_{c \in C} F_R(\downarrow c)}$, that is, there exists $D \in M(y)$ such that $D \subseteq \bigcup_{c \in C} cR$, since $\langle Y, \leq, M \rangle$ is direct.

(3) implies (2): if $F_R(S) = F_R(\overline{S})$ for all $S \in \mathcal{O}(X, \leq)$ and I_j are closed, then

$$F_R(\bigvee_{j \in J} I_j) = F_R(\overline{\bigcup_{j \in J} I_j}) = F_R(\bigcup_{j \in J} I_j) = \bigcup_{j \in J} F_R(I_j)\,. \qquad \square$$

Our next goal is to define a category whose objects are presentations and whose arrows are simulations. We first define it simply as a graph, that we shall be able to compare to the category of join-semilattices.

Definition 3.10. We let $\mathfrak{Sim}$ be the graph whose objects are direct presentations and whose arrows are simulations.

We let $\mathfrak{L}$ be the graph morphism that associates to a presentation $\langle X, \leq, M\rangle$ the lattice $\mathfrak{L}(X, \leq, M)$ and, to a simulation R from $\langle X, \leq, M\rangle$ to $\langle Y, \leq, M\rangle$, the join-homomorphism $F_R : \mathfrak{L}(X, \leq, M) \to \mathfrak{L}(Y, \leq, M)$.

Lemma 3.11. *$\mathfrak{L}$ is a full and faithful graph-morphism from $\mathfrak{Sim}$ to the reduct of the poset-enriched category of finite join-semilattices and join-preserving homomorphisms.*

As a matter of fact, the above Lemma is a restatement of Lemma 3.9. Next, let us denote by $\mathfrak{Latt}_\vee$ the poset-enriched category of finite join-semilattices and join-preserving homomorphisms.

Theorem 3.12. *Let us define the following additional structure on $\mathfrak{Sim}$:*

$$x\,\mathtt{id}\,y \;\; \textit{iff} \;\; y \in \overline{\downarrow x}\,, \qquad xS \cdot Tz \;\; \textit{iff} \;\; \exists C \in M(z) \;\; \textit{s.t.} \;\; C \subseteq \bigcup_{xSy} yT\,,$$

$$S \leq T \;\; \textit{iff} \;\; S \subseteq T\,.$$

Then $\mathfrak{Sim}$ is a poset-enriched category and $\mathfrak{L} : \mathfrak{Sim} \to \mathfrak{Latt}_\vee$ is a full and faithful poset-enriched functor.

Proof. It is enough to verify that the above structure correspond, along the bijection, to the poset-enriched strucure of $\mathfrak{Latt}_\vee$. Observe that the identity of $\langle X, \leq, M\rangle$ corresponds to the join homomorphism $\overline{(\,\cdot\,)} : \mathcal{O}(X, \leq) \to \mathfrak{L}(X, \leq, M)$, thus:

$$zR_{\mathtt{id}_{\langle X,\leq,M\rangle}}y \;\; \text{iff} \;\; y \in \overline{\downarrow x}\,.$$

Next, if S and T are composable simulations, then

$$\begin{aligned} xR_{F_T \circ F_S} z \;\; &\text{iff} \;\; z \in F_T(F_S(\overline{\downarrow x})) \\ &\text{iff} \;\; z \in F_T(F_S(\downarrow x)) \\ &\text{iff} \;\; z \in \overline{\bigcup_{xSy} yT} \\ &\text{iff} \;\; \exists C \in M(z) \;\; \text{s.t.} \;\; C \subseteq \bigcup_{xSy} yT\,. \end{aligned}$$

Finally:

$$\begin{aligned} F_S \leq F_T \;\; &\text{iff} \;\; F_S(U) \subseteq F_T(U)\,, \;\; \text{for all } U \in \mathcal{O}(X, \leq)\,, \\ &\text{iff} \;\; F_S(\overline{\downarrow x}) \subseteq F_T(\overline{\downarrow x})\,, \;\; \text{for all } x \in X \\ &\text{iff} \;\; xTy \text{ implies } xSy\,, \;\; \text{for all } x \in X \text{ and } y \in Y\,. \end{aligned}$$

□

3.2 Left adjoints in $\mathfrak{Sim}$ as oriented bisimulations

We identify lattice homomorphisms with those join-semilattice homomorphisms that preserve meets. As all the lattice under consideration are finite, hence complete, the property of an order preserving map to preserve meets is equivalent to that of being a right adjoint. Moreover, in any poset-enirched category right adjoints bijectively correspond to their left adjoints. The following folklore statement collects together the previous observations:

Lemma 3.13. *The category $\mathfrak{Latt}^{op}$, dual of the category of finite lattices and lattice homomorphisms, is isomorphic to the category of left adjoints in $\mathfrak{Latt}_\vee$.*

Recall that a pair $L \dashv R$ of adjoint 1-cells[4] in a poset-enriched category is defined by the unit and counit relations. Thus, the relations

$$\mathtt{id}_{\langle X,\leq,M\rangle} \subseteq L \cdot R\,, \tag{6}$$

$$R \cdot L \subseteq \mathtt{id}_{\langle Y,\leq,M\rangle}\,, \tag{7}$$

define left adjoints in $\mathfrak{Sim}$. Considering the poset isomorphism

$$\mathfrak{Sim}(\langle X,\leq,M\rangle,\langle Y,\leq,M\rangle) \simeq \mathfrak{Latt}_\vee(\mathfrak{L}(X,\leq,M),\mathfrak{L}(Y,\leq,M))$$

we immediately obtain the following statement:

Proposition 3.14. *The objectwise correspondence $\mathfrak{L}$ gives rise a contravariant full and faithful functor from the poset-enriched category of left adjoints in $\mathfrak{Sim}$ to the category $\mathfrak{Latt}$ of finite lattices.*

Our next goal, is to give an explicit characterization of left adjoints in $\mathfrak{Sim}$.

Lemma 3.15. *Let L be a simulation from $\langle X,\leq,M\rangle$ to $\langle Y,\leq,M\rangle$. If L has a right adjoint L^* within $\mathfrak{Sim}$, then L^* is defined as follows:*

$$yL^*x \quad \textit{iff} \quad xLy' \textit{ implies } D \ll \{\,y\,\} \textit{ for some } D \in M(y'). \tag{8}$$

Proof. Clearly, L is left adjoint to L^* if and only if F_L is left adjoint to F_{L^*}, whence

$$\begin{aligned}
yL^*x \quad &\text{iff} \quad x \in F_{L^*}(\downarrow y) = F_{L^*}(\overline{\downarrow y}) && \text{we assume } L^* \text{ is a simulation,}\\
&\text{iff} \quad \overline{\downarrow x} \subseteq F_{L^*}(\overline{\downarrow y})\\
&\text{iff} \quad F_L(\overline{\downarrow x}) \subseteq \overline{\downarrow y} && \text{by the adjointness relation,}\\
&\text{iff} \quad \text{for all } y',\ xLy' \text{ implies } D \ll \{\,y\,\},\ \text{for some } D \in M(y')\,.
\end{aligned}$$

□

In the following L^* shall denote the relation defined by (8). We shall not assume that L^* is itself a simulation, we shall only assume that L is a simulation.

Lemma 3.16. *L^* is a closed bimodule.*

[4] That is, L is left adjoint to R, and R is right adjoint to L.

Proof. We leave the reader to verify that L^* is a downset of $Y^{op} \times X$. We only verify that yL^* is closed, for an arbitrary $y \in Y$.

Thus we suppose that, for some $x \in X$ and $C \in M(x)$, yL^*c for each $c \in C$. The relations yL^*c amounts to $F_L(\downarrow c) \subseteq \overline{\downarrow y}$. Next, we have $x \in \overline{\downarrow C}$, whence

$$F_L(\downarrow x) \subseteq F_L(\overline{\downarrow C}) = F_L(\bigvee_{c \in C} \overline{\downarrow c}) = \overline{\bigcup_{c \in C} F_L(\downarrow c)} \subseteq \overline{\downarrow y}\,,$$

where we have used the fact that F_L preserves joins and that $F_L(\overline{\downarrow x}) = F_L(\downarrow x)$, for each $x \in X$. The relation so derived states that yL^*x. $\square$

Lemma 3.17. *If L^* is defined as in* (8)*, then the counit relation* (7) *holds whereas the unit relation* (6) *holds if and only if*

$$\exists C \in M(x) \ \ s.t. \ \ C \subseteq \bigcup_{xLy} yL^*\,. \tag{9}$$

Proof. Let us show that $yL^* \cdot Ly'$ implies $y' \in \overline{\downarrow y}$. There exists $D \in M(y')$ such that, for each $d \in D$, there exists x_d such that yL^*x_d and x_dLd. By (8), if x_dLy' then $D' \ll y$ for some $D' \in M(y')$. We have therefore $D'_d \ll y$, for some $D'_d \in M(d)$ and for each $d \in D$. Let $E \in M(y')$ be such that $E \ll \bigcup_{d \in D} D_d$, then $E \ll \{\, y \,\}$ so that $y' \in \overline{\downarrow y}$.

Condition (9) immediately follows from (6), considering that $x \in \overline{\downarrow x}$ and consequently $xL \cdot L^*x$. Thus, let us assume (9) and prove (6). Let $x' \in \overline{\downarrow x}$ so that, for some $C' \in M(x')$, $C' \ll \{\, x \,\}$. Let C be as in (9), then, for each $c' \in C'$ there exists $D_{c'} \in M(c')$ with $D_{c'} \ll C$. Next, let $E \in M(x')$ be such that $E \ll \bigcup_{c' \in C'} D_{c'}$, then $E \ll C$ and therefore $E \subseteq \bigcup_{xLy} yL^*$, witnessing that $xL \cdot L^*x'$. $\square$

3.3 A full and faithful functor

Next, we collect our previous observations into the main result of this Section.

Definition 3.18. A *presentation morphism*, or *oriented bisimulation*, from $\langle X, \leq, M \rangle$ to $\langle Y, \leq, M \rangle$ is a simulation L from $\langle X, \leq, M \rangle$ to $\langle Y, \leq, M \rangle$ such that

1. L^* is a simulation,
2. condition (9) holds: for each $x \in X$ there exists $C \in M(x)$ such that, for each $c \in C$, there exists y_c with xLy_c and such that cLy' implies $D \ll \{\, y_c \,\}$, for some $D \in M(y')$.

Proposition 3.19. *The identity simulation is an oriented bisimulation, and oriented bisimulations compose.*

Proof. The statement is an immediate consequence of the fact that the identity is a left adjoint, and that left adjoints compose. $\square$

For an oriented bisimulation from $\langle X, \leq, M \rangle$ to $\langle Y, \leq, M \rangle$, define the map $\mathfrak{L}_R : \mathfrak{L}(Y, \leq, M) \to \mathfrak{L}(X, \leq, M)$ as

$$\mathfrak{L}_R(S) = \overline{\{ x \in X \mid \exists y \in S \,\text{s.t.}\, y L^* x \}}\,.$$

Notice that $\mathfrak{L}_R = F_{R^*}$, thus in particular $\mathfrak{L}_R$ preserves joins, as R^* is a simulation. It also preserves meets as F_{R^*} is a right adjoint.

Theorem 3.20. *Let $\mathfrak{BSim}$ the category whose objects are presentations and whose arrows are oriented bisimulations. Then $\mathfrak{L}$ is a contravariant full and faithful functor from the category $\mathfrak{BSim}$ to the category $\mathfrak{Latt}$ of finite lattices.*

Proof. $\mathfrak{L}$ is faithful, as the covariant functor $\mathfrak{L} : \mathfrak{Sim} \to \mathfrak{Latt}_\vee$ is already faithful. It is full, as if $f : \mathfrak{L}(Y, \leq, M) \to \mathfrak{L}(X, \leq, M)$ is a lattice morphism, then it is a right adjoint in $\mathfrak{Latt}_\vee$ and has a left adjoint l within $\mathfrak{Latt}_\vee$. By the uniqueness of the adjoint, we necessarily have $f = \mathfrak{L}_{R_l^*}$. □

To end the discussion, let us introduce the following definition that shall occur often on the rest of the paper.

Definition 3.21. A presentation $\langle X, \leq, M \rangle$ is *subcanonical* if each downset $\downarrow x$ is closed.

4 From lattices to presentations

4.1 The OD-graph as an adjoint functor

We come back to finite lattices. We remark that the following definition – here formulated for finite lattices only – extend to lattices having the minimal join-cover refinement property.

Definition 4.1. The *OD-graph of a finite lattice* L, noted here $\mathcal{G}(L)$, is the structure $\langle J(L), \leq, \mathcal{M} \rangle$ where

- $J(L)$ is the set of join-irreducible elements of L,
- $\leq$ is the restriction of the order of L to $J(L)$,
- for each $j \in J(L)$, $\mathcal{M}(j)$ is the set of minimal join-covers of j.[5]

Lemma 4.2. *The OD-graph of a lattice L is a direct subcanonical presentation.*

[5]Let us recall that the name OD-graphs was introduced in [22] where the structure $\langle J(L), \leq, \mathcal{M}^{\text{nt}} \rangle$ was considered instead. Let us also recall the definition of the dependency relation between join-irreducible elements: jDk if $k \in C$ for some $C \in \mathcal{M}^{\text{nt}}(j)$. Thus it is easily recognized the meaning of OD, which stands for Order and Dependency of join-irreducible elements.

Proof. Let us verify that $\mathcal{G}(L)$ is direct, that is, it satisfies conditions (1)-(3) from Lemma 3.4. If $j \leq k$ and $D \in \mathcal{M}(k)$, then $j \leq k \leq \bigvee D$, whence $C \ll D$ for some $C \in \mathcal{M}(j)$. Also, we have $\{ j \} \in \mathcal{M}(j)$ and, of course, $\{ j \} \ll \{ j \}$. Next, let $C \in \mathcal{M}(j)$ and, for each $c \in C$, let $D_c \in M(c)$. Then $j \leq \bigvee \bigcup_{c \in C} D_c$, thus $E \ll \bigcup_{c \in C} D_c$ for some $E \in M(c)$.

Finally, let us verify that $\mathcal{G}(L)$ is subcanonical: if $D \in \mathcal{M}(k)$ and $D \ll \{ j \}$, then $\bigvee D \leq j$ and $k \leq \bigvee D \leq j$. □

The following Proposition, which is possibly part of the folklore of lattice theory, appears in [22, §2].

Proposition 4.3. *Let $\eta_L : L \to \mathfrak{L}(\mathcal{G}(L))$, where for $l \in L$ we have*

$$\eta_L(l) = \bigcup \{ j \in J(L) \mid j \leq l \} .$$

Then η_L is an isomorphism of lattices.

Proof. Let us verify first that $\eta_L(l)$ is closed. If $C \in \mathcal{M}(j)$ and $C \subseteq \eta_L(l)$, then $\bigvee C \leq l$, whence $j \leq \bigvee C \leq l$.

Next η_L is certainly order preserving. In order to argue that it is an isomorphism of lattices, it is enough to argue that it has on order preserving inverse. We let

$$\eta_L^{-1}(I) = \bigvee I ,$$

so that η_l^{-1} certainly is order preserving. We have

$$\begin{aligned} \eta_L(\bigvee I) &= \{ j \in J(L) \mid j \leq \bigvee I \} \\ &= \{ j \in J(L) \mid \exists C \in \mathcal{M}(j) \text{ s.t. } C \ll I \} = I , \end{aligned}$$

as I is closed, and $\mathcal{G}(L)$ is direct. On the other direction,

$$\bigvee \eta_L(l) = \bigvee \{ j \in J(L) \mid j \leq l \} = l ,$$

as usual. □

We next prove the following statement:

Proposition 4.4. *For each lattice morphism $f : L \to \mathfrak{L}(X, \leq, M)$ there exists a unique direct bisimulation $R : \langle X, \leq, M \rangle \to \mathcal{G}(L)$ such that $f = \mathfrak{L}_R \circ \eta_L$.*

Let us recall that the property stated in the Proposition, that we illustrate as usual with a diagram

$$\begin{tikzcd}
L \arrow[rr, "\eta_L"] \arrow[dr, "f"'] & & \mathfrak{L}(J(L), \leq, \mathcal{M}) \arrow[dl, dotted, "\mathfrak{L}_R"] & & \langle J(L), \leq, \mathcal{M} \rangle \\
& \mathfrak{L}(X, \leq, M) & & & \langle X, \leq, M \rangle \arrow[u, dotted, "\exists ! R"]
\end{tikzcd}$$

suffices to make $\mathcal{G}$ into a functor and to witness then the adjunction $\mathcal{G} \dashv \mathfrak{L} : \mathfrak{BSim}^{op} \to \mathfrak{Latt}$, i.e. the existence of a natural bijection

$$\mathfrak{Latt}(L, \mathfrak{L}(X, \leq, M)) \quad \simeq \quad \mathfrak{BSim}^{op}(\mathcal{G}(L), \langle X, \leq, M \rangle)\,.$$

Proof. The statement immediately follows from the fact that the functor $\mathfrak{L}$ is full and faithful and η_L invertible. Faithfulness implies uniqueness, and fullness implies existence: the oriented bisimulation R we are looking for is the unique one such that $\mathfrak{L}_R = f \circ \eta_L^{-1}$. □

Namely, we have

$$xRj \quad \text{iff} \quad j \leq \ell(\overline{\downarrow x}) \tag{10}$$

where $\ell \dashv f$.

For $f : L_0 \to L_1$, $\mathcal{G}(f)$ – the morphism part of the (contravariant) functor $\mathcal{G}$ – is then defined as the unique R such that

$$\begin{tikzcd}
L_0 \arrow[r, "\eta_{L_0}"] \arrow[d, "f"] & \mathfrak{L}(J(L_0), \leq, \mathcal{M}) \arrow[d, dotted, "\mathfrak{L}_R"] & \langle J(L_0), \leq, \mathcal{M} \rangle \\
L_1 \arrow[r, "\eta_{L_1}"] & \mathfrak{L}(J(L_1), \leq, \mathcal{M}) & \langle J(L_1), \leq, \mathcal{M} \rangle \arrow[u, dotted, "\exists ! R"']
\end{tikzcd}$$

An explicit computation again gives

$$k\mathcal{G}(f)j \quad \text{iff} \quad j \leq \ell(k)\,,$$

with $\ell \dashv f$.

4.2 The duality theorem

Let us recall the following fact. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors, and consider the full subcategories of $\mathcal{D}$, resp. of $\mathcal{D}$, determined by the objects whose unit, resp. the counit, are invertible arrows. Then F, G induce an equivalence between these full subcategories.

Therefore, for the adjunction $\mathcal{G} \dashv \mathfrak{L} : \mathfrak{BSim}^{op} \to \mathfrak{Latt}$, we want to determine these full subcategories. This problem turns out to have an simple solution.

We have already observed that, for each lattice L, the unit of the adjunction $\eta_L : L \to \mathfrak{L}(J(L), \leq, \mathcal{M})$ is invertible. However, the counit of the adjunction

$$E_{\langle X, \leq, M \rangle} : \langle X, \leq, M \rangle \to \mathcal{G}(\mathfrak{L}(X, \leq, M))$$

is invertible as well. To see this, recall that

$$\mathfrak{L}(E_{\langle X,\leq,M\rangle}) : \mathfrak{L}(\mathcal{G}(\mathfrak{L}(X,\leq,M))) \to \mathfrak{L}(X,\leq,M)$$

has $\eta_{\mathfrak{L}(X,\leq,M)}$ as inverse map and that $\mathfrak{L}$ is full and faithful functor. As a matter of fact, it is a standard Lemma of elementary category theory, cf. [21, §IV.3], that a right adjoint is full and faithful if and only if the counit is invertible.

Thus, we obtain the following Theorem:

Theorem 4.5. *The functors $\mathcal{G}, \mathfrak{L}$ are inverse parts of an equivalence of categories.*

5 Monic oriented bisimulations

In this Section we focus on a restricted duality, namely the one we obtain when considering epic arrows in the category $\mathfrak{Latt}$ and, respectively, monic arrows in $\mathfrak{BSim}$. The reason for focusing pausing on this duality possibly is historical: the first results on the OD-graph of a lattice concern the representation of congruences of a finite lattice, see [22, Lemma 1.2] and [10, §?]. Also, it was already observed in [18, §3] that a nice duality for lattices arise if we consider only epimorphisms.

Our interest in developing a full duality theory for finite lattices stems from the problem of computing, given the OD-graph of a lattice, the OD-graph of some quotient. This problem has a very simple solution, a fact that we shall illustrate by computing of the OD-graph of the Associahedra from the OD-graph of the Permutohedra.

5.1 Reworking oriented bisimulations

The conditions – see Definition 3.18 – that characterize left adjoints in $\mathfrak{Sim}$ and that define oriented bisimulation might appear, at a first sight, obscure and difficult to work with. Yet, these conditions might look – depending on the taste of the reader – more elegant if we replace ideals with their maximal antichains. Also, the same conditions considerably simplify if we assume that each $x \in X$ gives rise to a closed downset $\downarrow x$, that is if a presentation is subcanonical. For example, for a subcanonical presentation $\langle Y, \leq, M\rangle$ formula (8) simplifies to

$$yL^*x \quad \text{iff} \quad xLy' \,\text{implies}\, y' \leq y\,. \tag{11}$$

Definition 5.1. Let P, Q be two posets. Call a function $f : P \to \mathcal{P}(Q)$ a *layer* from P to Q if, for each $p, p' \in P$,

1. $f(p)$ is an antichain,
2. $p \leq p'$ implies $f(p) \ll f(p')$.

It is easily seen that the transformations

$$pR^f q \text{ iff } q \in \downarrow f(q)\,, \qquad\qquad q \in f^R(p) \text{ iff } q \in \max pR\,,$$

are inverse bijections between bimodules and layers from P to Q. Observe in particular that, under the assumption of subcanonicity,

$$xLy \text{ iff } \{\, y \,\} \ll f^L(x)\,, \qquad\qquad yL^*x \text{ iff } f^L(x) \ll \{\, y \,\}\,,$$

where L^* is defined as in (11).

Definition 5.2. Let $\langle X, \leq, M\rangle$ and $\langle Y, \leq, M\rangle$ be two subcanonical presentations. Say that a layer function from $\langle X, \leq\rangle$ to $\langle Y, \leq\rangle$ is *p-morphic* if it satisfies the following conditions :

1. $C \in M(y)$ and $C \ll f(x)$ implies $\{\, y \,\} \ll f(x)$,
2. for all $x \in X$ there exists $C \in M(x)$ and a collection $\{\, y_c \mid c \in C \,\}$ such that $f(c) \ll \{\, y_c \,\}$ and $\{\, y_c \,\} \ll f(x)$,
3. $\{\, y \,\} \ll f(x)$ and $C \in M(x)$ implies $D \ll \bigcup_{c \in C} f(c)$ for some $D \in M(y)$,
4. if $f(x) \ll \{\, y \,\}$ and $D \in M(y)$, then for some $C \in M(x)$ and for each $c \in C$ there exists $d \in D$ such that $f(c) \ll \{\, d \,\}$.

We just rephrased the conditions for a bimodule to be an oriented bisimulation, using the correspondence that transforms a bimodule to a layer function. Thus the following proposition should not come as a surprise:

Proposition 5.3. *Let $\langle X, \leq, M\rangle$ and $\langle Y, \leq, M\rangle$ be two subcanonical direct presentations. Oriented bisimulations and p-morphic layer functions from $\langle X, \leq, M\rangle$ to $\langle Y, \leq, M\rangle$ are in a bijective correspondence.*

We start next making some observations towards a simplification of these conditions. Condition 3 is actually equivalent to the simpler

3' $y \in f(x)$ and $C \in M(x)$ implies $D \ll \bigcup_{c \in C} f(c)$ for some $D \in M(y)$.

As a matter of fact, if $y' \leq y \in f(x)$ and $D \in M(y)$ is such that $D \ll C$, then there exists $D' \in M(y')$ such that $D' \ll D$, whence $D' \ll \bigcup_{c \in C} f(c)$.

Next, let us suppose that $f(x) = \{\, \tilde{f}(x) \,\}$ is a singleton. Then condition 3 is equivalent to

3". $C \in M(x)$ implies $D' \ll \tilde{f}(C)$ for some $D' \in M(\tilde{f}(x))$.

and condition 4 is equivalent to

4". $D \in M(\tilde{f}(x))$ implies $\tilde{f}(C') \ll D$ for some $C' \in M(x)$.

We suppose next that $f(x) = \{ \tilde{f}(x) \}$ is a singleton and $\tilde{f}$ is an embedding. Conditions $3''$ and $4''$ imply $\tilde{f}(C') \ll \tilde{f}(C)$, whence $C' \ll C$ and $C = C'$ if $M(x)$ is an antichain. If D' is an antichain, then $\tilde{f}(C) \ll D' \ll \tilde{f}(C)$ implies $C = \tilde{f}(D')$. Under similar conditions we have $D' \ll D$, $D' = D$, and $D \ll f(C') \ll D$, whence $f(C') = D$. We have therefore that $3''$ and $4''$ are equivalent to

3-4. $D \in M(\tilde{f}(x))$ if and only if $D = \tilde{f}(C)$ for some $C \in M(x)$.

We stress at this point the similarity – but also the difference – with the notion of bisimulation arising from monotone logic and coalgebra theory [16]. Yet, in order to have a sort of bisimulation as understood there, we need $\{ y \} <\!\!< f(x)$ in place of $f(x) \ll \{ y \}$ in condition 4.

5.2 The restricted duality

Recall that a map f in a category is epic if $g \circ f = h \circ f$ implies $g = h$. Surjective homomorphisms of algebras are epic, there exists epic maps of algebra that are not surjective. The two notions, epimorphism and epic map, coincide on the category of finite lattices.

Lemma 5.4. *If $f : L \to M$ is an epic map in the category $\mathfrak{Latt}$ of finite lattices, then f is a surjective homomorphism, whence a regular epic.*

Proof. If $\ell \dashv f$ then, by the usual properties of adjunctions, $f \circ \ell \circ f = f$. As f is epic, then $f \circ \ell = \mathtt{id}_M$. That is, f is split epic as an order preserving map and in particular it is a surjective function. As usual, this implies that it is the coequalizer of its kernel pair. □

The following lemma and considerations are analogous of [12, Proposition 3.4] for the theory of generalized Kripke frames.

Lemma 5.5. *Let $g : L_1 \to L_0$ be a lattice morphism and let $\ell \dashv g$. Then g is epic if and only if ℓ is an embedding and, in this case, ℓ restricts to an embedding from $J(L_0) \to J(L_1)$.*

Proof. If g is epic in the category $\mathfrak{Latt}$ and has a left adjoint ℓ, then the relation $g \circ \ell \circ g = g$, that holds for adjoint pairs, we deduce that $g \circ \ell = id_{L_0}$, so that ℓ is split monic, whence an embedding. A similar argument shows that if ℓ is an embedding, then $\ell \circ g = id_{L_1}$.

Suppose next that $\ell(j) = x \vee y$, then $j = g(\ell(j)) = g(x) \vee g(y)$ and – say – $j = g(x)$. Hence $\ell(j) = \ell(g(x)) \leq x$. The relation $\ell(j) = x \vee y$ implies on the other hand that $x \leq \ell(j)$, whence $\ell(j) = x$. A similar argument shows that $\ell(j) \neq \perp$, as $j \neq \perp$. □

Definition 5.6. A bimodule R from $\langle X, \leq \rangle$ to $\langle Y, \leq \rangle$ is said to be *representable* if there exists a function $f : \langle X, \leq \rangle \to \langle Y, \leq \rangle$ such that

$$xRy \quad \text{iff} \quad y \leq f(x) \,.$$

Observe that from the definition it follows that such an f is order preserving: if $x \leq x'$, then from $xRf(x)$ we deduce $x'Rf(x)$, whence $f(x) \leq f(x')$.

Let us say that $\langle X, \leq, M \rangle$ is *reduced* if it is subcanonical and each $\overline{\downarrow x} = \downarrow x$ is join-irreducible in $\mathfrak{L}(X, \leq, M)$.

Lemma 5.7. *Let $\langle X, \leq, M \rangle$, $\langle Y, \leq, M \rangle$ be reduced presentations. If $R : \langle X, \leq, M \rangle \to \langle Y, \leq, M \rangle$ is a monic arrow in $\mathfrak{Sim}$, then R is representable by an embedding f.*

Proof. Since the functor $\mathfrak{L} : \mathfrak{BSim}^{op} \to \mathfrak{Latt}$ is full and faithful R is monic if and only if $\mathfrak{L}_R$ is an epimorphism. As $F_R \dashv \mathfrak{L}_R$, then the previous Lemma ensures that F_R is an embedding and sends join-irreducible elements to join-irreducible elements. Recall that join-irreducible elements in $\mathfrak{L}(X, \leq, M)$ are exaclty those of the form $\downarrow x$ for some $x \in X$.

Thus, for $x \in X$ then we can find $f(x) \in Y$ such that $F_R(\downarrow x) = \downarrow f(x)$. Observe that f is an embedding, as F_R is an embedding. We obtain

$$xRy \quad \text{iff} \quad y \in F_R(\overline{\downarrow x}) \quad \text{iff} \quad y \in \downarrow f(x) \quad \text{iff} \quad y \leq f(x) \,. \qquad \square$$

Definition 5.8. Let $\langle X, \leq, M \rangle$ and $\langle Y, \leq, M \rangle$ be completely reduced. Say that an embedding $f : \langle X, \leq, M \rangle \to \langle Y, \leq, M \rangle$ is p-morphic if $D \in M(f(x))$ if and only if there exists $C \in M(x)$ such that $f(C) = D$.

Clearly, p-morphic embeddings compose and the identity is such an embedding, thus they form a category.

Theorem 5.9. *The category of completely reduced presentations and p-morphic embeddings and of lattices and epimorphism are dual to each other.*

5.3 From Permutohedra to Associahedra

In the following we shall denote by $\mathcal{P}_n$ the lattice of permutations on n elements. This lattice is known as the Permutohedron on n letters. We shall denote by $\mathcal{T}_n$ be n-th Tamari lattice, or Associahedron on $n+1$ letters. Elements of this lattice are finite binary trees with n internal nodes and $n+1$ leaves. We refer the reader to [3] for an introduction to these lattices.

According to [5, Theroem 9.6], given a permutation $\sigma \in \mathcal{P}_n$, we can construct a binary tree $\pi_n(\sigma) \in \mathcal{T}_n$ according to the following rules. If $n = 0$, then σ is the identity permutation, and $\pi_0(\sigma)$ is the unique binary tree with one leave. Let us suppose $n > 0$, consider the letter σ_n, and let

$$L = \{\, i \mid \sigma_i < \sigma_n \,\}\,, \qquad\qquad R = \{\, j \mid \sigma_n < \sigma_j \,\}\,.$$

Let now

$$\begin{aligned}
\sigma_l :& \{\, 1, \ldots, \sigma_n - 1 \,\} \xrightarrow{op} L \xrightarrow{\sigma|_L} \{\, 1, \ldots, \sigma_{n-1} \,\}\,, \\
\sigma_r :& \{\, 1, \ldots, n - \sigma_n \,\} \xrightarrow{op} R \xrightarrow{\sigma|_R} \{\, \sigma_n + 1, \ldots, n \,\} \xrightarrow{op} \{\, 1, \ldots, n - \sigma_n \,\}\,,
\end{aligned}$$

where *op* are the unique order preserving bijections. We obtain $\pi_n(\sigma)$ by grafting together (in the order) the two trees $\pi_{\sigma_n - 1}(\sigma_l)$ and $\pi_{n-\sigma_n}(\sigma_r)$ into one root. It was observed in [5] that π_n is order preserving; the following fact was made explicit in [24].

Proposition 5.10. *The order preserving maps $\pi_n : \mathcal{P}_n \to \mathcal{T}_n$ are lattice epimorphisms.*

A permutation σ is 312-*free* if there is no $i < j < k$ such that $\sigma_j < \sigma_k < \sigma_i$. Let ℓ_n denote the left adjoit to π_n, $\ell_n \dashv \pi_n$. It is known that, given a binary tree $t \in \mathcal{T}_n$, $\ell_n(t)$ is to be the unique 312-free permutation σ such that $\pi_n(\sigma) = t$.

Let us come to the the OD-graph of the lattice $\mathcal{P}_n$. We recall that a permutation σ is join-irreducible in $\mathcal{P}_n$ if and only if it has only one descent, i.e. we can write $\sigma = \sigma_1 \ldots \sigma_j \sigma_{j+1} \ldots \sigma_n$, where $\sigma_{j+1} < \sigma_j$ and all the other contiguous letters are in the right order. In [26] we observed that join-irreducible permutations are in bijection with quadruples (a, b, D_a, D_b) such that

- $a, b \in \{ 1, \ldots, n \}$ and $a < b$,
- $\{ D_a, D_b \}$ is a binary partition of the open interval (a, b).

Such a quadruple corresponds to the join-irreducible permutation

$$1, \ldots a - 1 \vec{D_a} b a \vec{D_b} b + 1 \ldots n$$

where $\vec{D_i}$ is the ordered list of elements in D_i. Using such a representation we were able to characterize the dependency relation D between join-irreducible elements:

$$\begin{aligned} &(a, b, D_a, D_b) D (c, d, D_c, D_d) \ \text{ iff} \\ &\qquad [c, d] \subsetneq [a, b],\ D_c = D_a \cap (c, d)\,, \ \text{ and } D_d = D_b \cap (c, d)\,. \end{aligned}$$

It is not difficult to generalize the results of [26] to obtain the following Proposition.

Proposition 5.11. *The OD-graph of the lattice $\mathcal{P}_n$ is isomorphic to the presentation $\langle X_n, \leq, M \rangle$ where*

- *an element of X is a quadruple (a, b, D_a, D_b) as before,*
- *$(a, b, D_a, D_b) \leq (c, d, D_c, D_f)$ if and only if*

  $$I(a, b, D_a, D_b) \subseteq I(c, d, D_c, D_f)\,,$$

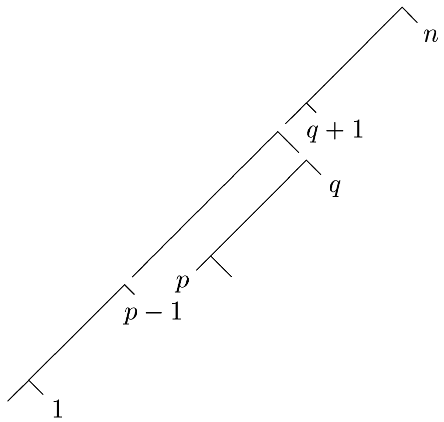
  where $I(a, b, D_a, D_b) = \{ (x, y) \mid x > y,\ x \in D_a \cup \{ b \},\ y \in D_b \cup \{ a \} \}$.
- *$C \in M(a, b, D_a, D_b)$ iff C is of the form*

  $$\{ (x_i, x_{i+1}, D_a \cap (x_i, x_{i+1}), D_b \cap (x_i, x_{i+1})) \mid i = 0, \ldots, k - 1 \}$$

  with $a = x_0 < x_i < \ldots x_{k-1} < x_k = b$.

Lemma 5.12. *An element (a, b, D_a, D_b) of X_n codes a 312-free permutation if and only if $D_b = \emptyset$.*

In order to compute the OD-graph of the Tamari lattice $\mathcal{T}_n$, we start by recalling what are its join-irreducible elements. Then we shall argue how they are sent by ℓ_n to such a quadruple. Join-irreducible elements in $\mathcal{T}_n$ are of the form $[p, q]$ with $1 \leq p < q \leq n$, where $[p, q]$ is the tree



As it is easily seen that

$$\pi_n(p, q, (p, q), \emptyset) = [p, q]\,,$$

we have

$$\ell_n([p, q]) = (p, q, (p, q), \emptyset)\,.$$

Combining this observation, Proposition 5.11, and Theorem 5.9, we deduce:

Theorem 5.13. *The OD-graph of the Tamari lattice $\mathcal{T}_n$ is isomorphic to the presentation $\langle Y_n, \leq, M \rangle$, where*

- $Y_n = \{\, [p, q] \mid 1 \leq p < q \leq n \,\}$,
- $[p, q] \leq [p', q']$ *if and only if* $[p, q] \subseteq [p', q']$ *(as intervals),*
- $C \in M([p, q])$ *if and only if* C *is of the form* $\{\, [x_i, x_{i+1}] \mid i = 0, \ldots, k-1 \,\}$ *where* $p = x_0 < x_1 < \ldots < x_{k-1} < x_k = q$.

6 Some representation theorems

The duality Theorem 4.5 is somewhat unsatisfying: given a direct presentation $\langle X, \leq, M \rangle$ the conunit of the adjunction

$$E : \langle X, \leq, M \rangle \to \mathcal{G}(\mathfrak{L}(X, \leq, M))$$

is an invertible map in the category $\mathfrak{BSim}$. Yet, the counit is far from being an isomorphism of structures, thus it is far from telling us what it means for a relational structure to be an OD-graph in any standard way. In this section we tackle this question and introduce some constructions of OD-graphs of atomistic lattices. These constructions shall open the way to the results of the next section.

6.1 The general representation theorem

We have already seen that $\mathcal{G}(L)$ is a directed subcanonical presentation. Let us list some of the properties of a presentation $\langle X, \leq M \rangle$ that in particular hold in a presentation of the form $\mathcal{G}(L)$. By construction, these are:

1. $\{ x \} \in \mathcal{M}(x)$, for each $x \in X$,
2. C is an antichain (w.r.t. $\leq$), for each $x \in X$ and $C \in \mathcal{M}(x)$,
3. $\mathcal{M}(x)$ is an antichain, for each $x \in X$.

Theorem 6.1. *If $\langle X, \leq, M \rangle$ is a directed subcanonical presentation satisfying conditions (1-3) above, then the principal ideal $\downarrow$ is an isomorphism of relational structures from $\langle X, \leq, M \rangle$ to $\mathcal{G}(\mathfrak{L}(X, \leq, M))$ that represents the counit of the adjunction.*

Proof. Let us remind first the formula for computing the closure of a downset Y of X:

$$\overline{Y} = \{ z \in X \mid \exists C \in M(z) \text{ s.t. } C \ll Y \}, \tag{12}$$

which holds as usual for directed presentations.

Since $\langle X, \leq, M \rangle$ is subcanonical, each $\downarrow x$ is closed. Let us therefore verify that the correspondence sending $x \in X$ to $\downarrow x$ gives rise to an isomorphism of structures.

We observe first that every join-irreducible element in $\mathfrak{L}(X, \leq, M)$ is of the form $\downarrow x$ for some $x \in X$. This follows from the fact that elements of this form join-generate the lattice $\mathfrak{L}(X, \leq, M)$: if I is a closed downset, then

$$I = \bigvee_{x \in I} \downarrow x \, .$$

Next, let us show that each $\downarrow x$ is join-irreducible. Let us suppose, therefore, that $\downarrow x = \bigvee_{i=1,\ldots,n} \downarrow y_i$. From

$$\downarrow x \subseteq \bigvee_{i=1,\ldots,n} \downarrow y_i = \overline{\bigcup \{ \downarrow y_i \mid i = 1, \ldots, n \}}$$

we derive that $C \ll \{ y_i \mid i = 1, \ldots, n \}$ for some $C \in M(x)$. From $\bigvee_{i=1,\ldots,n} \downarrow y_i \subseteq \downarrow x$, we derive $C \ll \{ y_i \mid i = 1, \ldots, n \} \ll \{ x \}$. Since bot $\{ x \}$ and C

belong to $M(x)$ which is an antichain, the relation $C \ll \{ x \}$ implies $C = \{ x \}$. Whence, from

$$\{ x \} \ll \{ y_i \mid i = 1, \ldots, n \} \ll \{ x \}$$

we derive that, for some $i = 1, \ldots, n$, $x \leq y_i \leq x$, that is $x = y_i$ for some i.

We have established up to know that the principal ideal map $\downarrow$is a bijection from X to $J(\mathfrak{L}(X, \leq, M))$; clearly it is also an embedding. To establish that it is an isomorphism of relational structures, we are left to prove that $D \in \mathcal{M}(\downarrow x)$ if and only if $D = \{ \downarrow c \mid c \in C \}$ for some $C \in M(x)$. We shall do that by characterizing the minimal join-covers of $\downarrow x$ as being those of the form $\{ \downarrow c \mid c \in C \}$ for some $C \in M(x)$.

If $C \in M(x)$ then

$$\downarrow x \subseteq \overline{\downarrow C} = \bigvee_{c \in C} \downarrow c \,,$$

so that $\{ \downarrow c \mid c \in C \}$ is a cover of $\downarrow x$. Next, let us suppose that $\{ \downarrow y \mid y \in Y \}$ is a cover of $\downarrow x$:

$$\downarrow x \subseteq \bigvee_{y \in Y} \downarrow y \,, \quad \text{that is} \quad x \in \overline{\downarrow Y} \,.$$

By (12) there exists $C \in M(x)$ such that $C \ll Y$. As $\downarrow$is an embedding, we have

$$\{ \downarrow c \mid c \in C \} \ll \{ \downarrow y \mid y \in Y \} \,.$$

This shows that

$$\{ \{ \downarrow c \mid c \in C \} \mid C \in M(x) \}$$

are exactly the minimal join-covers of x in the lattice of closed downsets.

Finally, recall that the counit corresponds under the bijection of homsets to the identity of $\mathfrak{L}(X, \leq, M)$. Thus, formula (10) gives that, for a join-irreducible element I of $\mathfrak{L}(X, \leq, M)$,

$$x E_{\langle X, \leq, M \rangle} I \quad \text{iff} \quad I \subseteq \overline{\downarrow x} \,.$$

As $\langle X, \leq, M \rangle$ is subcanonical and each $\downarrow x$ is join-irreducible, we obtain that the bimodule E is represented by $\downarrow$ $\square$

6.2 Atomistic lattices and atomistic presentations

Let us recall that a lattice is said to be *atomistic* if every element is the join of the atoms below it. Equivalently, L is atomistic if $J(L) = At(L)$, where $At(L)$ is the set of atoms of L. We examine next how the representation Theorem 6.1 restricts to atomistic lattices. Morally, we generalize the characterization, up to isomorphism, of a join-dependency relation [15, Theorem 3.1]. The proof of this result takes advantage of atomistic lattices and has been one of the staring point for the present investigation.

In the following Δ_X shall denote the identity relation of (i.e. the discrete order on) the set X.

Definition 6.2. An atomistic presentation is a pair $\langle X, M \rangle$ with $\langle X, \Delta_X, M \rangle$ a presentation. If L is an atomistic lattice then $\mathcal{G}(L) = \langle At(L), \Delta_{At(L)}, \mathcal{M} \rangle$, so that we shall say that $\langle At(L), \mathcal{M} \rangle$ is its atomistic presentation.

We begin pointing out some properties of the atomistic presentation of an atomistic lattice.

Lemma 6.3. *If L is an atomistic lattice, then its presentation $\langle At(L), \mathcal{M} \rangle$ satisfies the following conditions, for each $a \in At(L)$:*

1. $\{ a \} \in \mathcal{M}(a)$,
2. $\mathcal{M}(a)$ *is an antichain w.r.t. subset inclusion,*
3. *if $C \in \mathcal{M}(a)$ and $C \neq \{ a \}$, then C is not a singleton,*
4. *if $C \in \mathcal{M}(a)$, $c \in C$, and $D \in \mathcal{M}(c)$, then there exists $E \in \mathcal{M}(a)$ such that $E \subseteq (C \setminus \{ c \}) \cup D$.*

Proposition 6.4. *If an atomistic presentation $\langle A, M \rangle$ satisfies conditions 1-4 of Lemma 6.3, then it is isomorphic to the presentation of an atomistic lattice.*

The proof of this result is straightforward, given Theorem 6.1, and therefore we omit it.

Example 6.5. We come next to an handy way to construct atomistic presentations that shall be used later in Section 7. Let us say that an atomistic presentation $\langle A, M \rangle$ is *pointed* if it comes with a specified element $a \in A$; we shall write then $\langle A, M, a \rangle$. Let $\langle A_i, M_i, a_i \rangle$, $i \in I$, be pointed atomistic presentations. Let also $\mathcal{F} \subseteq \mathcal{P}(I)$ be an antichain such that $\#X \geq 2$, for each $X \in \mathcal{F}$. If $X \subseteq I$, then we say that $C : X \to \bigcup_{x \in X} M(a_x)$ is a *choice for X* if $C(x) \in M(a_x)$ for each $x \in X$. We define a new pointed atomistic presentation

$$\langle A, M, a_0 \rangle = \bigoplus_{a_0, \mathcal{F}} \langle A_i, M_i, a_i \rangle$$

as follows:

$$A = \{ a_0 \} \cup \biguplus_{i \in I} A_i$$

$$M(a) = \begin{cases} M_i(a) \,, & a \in A_i \\ \{ \{ a_0 \} \} \cup \{ \bigcup_{x \in X} C(x) \mid X \in \mathcal{F}, C \text{ a choice for } X \} \,, & a = a_0 \,. \end{cases}$$

Observe that $\{ a_x \mid x \in X \} \in M(a_0)$, for each $X \in \mathcal{F}$: indeed, the function sending x to $\{ a_x \}$ always is a choice for X.

Proposition 6.6. *$\bigoplus_{a_0,\mathcal{F}}\langle A_i, M_i, a_i\rangle$ is an atomistic presentation which satisfies conditions 1-4, whenever the presentations $\langle A_i, M_i, a_i\rangle$, $i \in I$, satisfy these conditions.*

Proof. Only condition 4 is not self-evident. Moreover, it will be enough to verify it for $C \in M(a_0)$ and $C \neq \{ a_0 \}$.

Let $X \in \mathcal{F}$ and $C : X \to \bigcup_{x\in X} M(a_x)$ be a choice for X, so that $\bigcup_{x\in X} C(x) \in M(a_0)$. Let $d \in C(x)$, for some $x \in X$, and let $D \in M(d)$. Then, for some $E \in M(a_x)$, $E \subseteq C(x) \setminus \{ d \} \cup D$. Define then $C'(x) = E$ and $C'(y) = C(y)$ if $y \neq x$. Then C' is a choice for X and

$$\bigcup_{x\in X} C'(x) \subseteq ((\bigcup_{x\in X} C(x)) \setminus \{ d \}) \cup D\,. \qquad \square$$

Example 6.7. Let $\langle T, \leq\rangle$ be a finite tree such that each internal note has at least two sons. A maximal antichain in T is a subset $A \subseteq T$ such that for each $t \in T \setminus A$, either $A \cup \{ t \}$ is not an antichain. We can define an atomistic presentation $\langle T, M\rangle$, satisfying conditions *1-4*, as follows:

$$M(t) = \{ A \subseteq \uparrow t \mid A \text{ is a maximal antichain of } \uparrow t \}\,.$$

Of course, $\langle T, M\rangle$ can be generated from a sequence of the operation $\oplus$ starting from the atomistic presentation $\langle\{ * \}, M\rangle$ with $M(*) = \{ \{ * \} \}$.

7 Interpretations, games, fixed-points

In this Section we first show how the duality gives rise to a semantics of lattice terms into presentations. The semantics that we develop turns out to be a covering semantics in the sense of [13]. We also propose a game theoretic account of this semantics. The actual goal of the Section is to exhibit the use of the duality, of the semantics, and of all the tools developed so far. In particular, we shall construct ad-hoc models to falsify some lattice equation. The problem that we tackle stems from fixed-point theory [6, 2] on lattices [25]. It is well known [1] that extremal fixed-point terms are redundant on distributive lattices: for any lattice polynomial ϕ, the equation $\phi(\perp) = \phi^2(\perp)$ holds on every distributive lattice and shows that the term $\mu_x.\phi$, meant to denote the least fixed-point of ϕ, can be replaces by the simpler term $\phi(\perp)$. We propose a generalization of this fact. We consider finite lattices in the varieties $\mathcal{L}_n$, whose members have the property that every sequence of the join-dependency relation has length at most n. Here the equation $\phi^{n+1}(\perp) = \phi^{n+2}(\perp)$ holds and exhibits $\phi^{n+1}(\perp)$ as the least fixed-point of ϕ. The generalization is non-trivial, as we show that $\frac{n-2}{3}$ is a lower bound for the integers i such that $\phi^i(\perp)$ is a fixed-point of ϕ on $\mathcal{L}_n$.

7.1 A covering semantics for lattice terms

We consider next interpretations of terms (and of equations) into lattices of directed presentations of the form $\mathcal{Y} = \langle Y, \leq, M\rangle$. Let X be a set of variables, denote by $\mathcal{T}(X)$ the algebra of lattice terms whose free variables are among X, denote by $\mathcal{F}(X)$ the free lattice generated by the set X. Recall that there are natural bijections between the following type of data: valuations $v : X \to \mathfrak{L}(Y, \leq, M)$, algebra morphisms $v' : \mathcal{T}(X) \to \mathfrak{L}(Y, \leq, M)$ such that $v'(x) = x$, lattice homomorphisms $\tilde{v} : \mathcal{F}(X) \to \mathfrak{L}(Y, \leq, M)$ such that $\tilde{v}(x) = v(x)$. In order to simplify the notation, we use shall often use the same notation v for the three different kind of data.

We introduce next a semantical relation reminiscent of Kripke semantics in modal logics. This kind of semantics has already been considered for linear logic [13] and topos theory [7].

Definition 7.1. For $v : X \to \mathfrak{L}(Y, \leq, M)$, $y \in Y$ and $t \in \mathcal{T}(X)$, the relation $y \models_v t$ is defined inductively on t as follows :

- $y \models_v x$ if $y \in v(x)$,
- $y \models_v \bigwedge_{i \in I} t_i$ if $y \models t_i$ for each $i \in I$,
- $y \models_v \bigvee_{i \in I} p_i$ if there exists $C \in M(y)$ such that, for all $c \in C$ there exists $i \in I$ with $c \models_v t_i$.

The following result is quite obvious:

Lemma 7.2. *$y \models_v t$ if and only if $y \in v(t)$.*

The same relation is characterized by means of a game $\mathcal{G}(\mathcal{Y}, v, t)$ between two players, Eva and Adam. The set of positions of this game includes the Cartesian product of Y with the set of subterms of t, but is not restricted to it. The description of the game follows:

- In position (y, x) Eva wins if $y \in v(x)$ and otherwise Eva loses.
- In position $(y, \bigwedge_{i \in I} t_i)$ Adam chooses $i \in I$ and moves to (y, t_i).
- In position $(y, \bigvee_{i \in I} t_i)$ Eva chooses $C \in M(y)$ and moves to $(C, \bigvee_{i \in I} t_i)$; from here Adam chooses $c \in C$ and moves to $(c, \{ t_i \mid i \in I \})$; from here Eva chooses $i \in I$ and moves to (c, t_i).

If a player cannot move then he loses. We have

Lemma 7.3. *Eva has a winning strategy from (y, t) in the game $\mathcal{G}(\mathcal{Y}, v, t)$ if and only if $y \in v(t)$, if and only if $y \models_v t$.*

7.2 Varieties of lattices with trivial least fixed-points

Let us recall the definition of the join-dependency relation D:

$$j D k \,\text{iff}\, j \neq k \text{ and } k \in C, \text{ for some } C \in \mathcal{M}(j)\,.$$

We let $\mathcal{L}_n$ be the class of all finite lattices such that every sequence

$$j_0 D j_1 D \ldots D j_k$$

has length k which is at most n. By [28, Section 3] (see also [22] as well as the next Section) this is an equational class, in the following sense: there exist equations D_{n+1}, $n \geq 0$, such that a finite lattice L satisfies D_{n+1} if and only if $L \in \mathcal{L}_n$. Consider that $\mathcal{L}_0$ is the class of finite distributive lattices and that $\mathcal{L}_n$ is a class of finite lattices obeying a weak form of distributivity.

Next, let ϕ be a lattice term containing the variable z. Define terms $\phi^n(\bot)$, $n \geq 0$, as follows

$$\phi^0(\bot) = \bot\,, \qquad\qquad \phi^{n+1}(\bot) = \phi[\,\phi^n(\bot)/z\,]\,.$$

Proposition 7.4. *The equation*

$$\phi^{n+2}(\bot) = \phi^{n+1}(\bot)$$

holds on $\mathcal{L}_n$.

Proof. Clearly, the inequation $\phi^{n+1}(\bot) \leq \phi^{n+2}(\bot)$ always holds, so that we only need to verify that the inequation $\phi^{n+2}(\bot) \leq \phi^{n+1}(\bot)$ holds as well. To this goal, let $L \in \mathcal{L}_n$ and consider a valuation $v : X \to \mathfrak{L}(J(L), \leq, \mathcal{M})$; we suppose that $j_0 \models_v \phi^{n+2}(\bot)$ and prove that $j_0 \models \phi^{n+1}(\bot)$. Let $\mathcal{Y} = \langle J(L), \leq, \mathcal{M}\rangle$, we consider a winning strategy for Eva in the game $\mathcal{G}(\mathcal{Y}, v, \phi^{n+2}(\bot))$ from $(j_0, \phi^{n+2}(\bot))$ and transform it into a winning strategy for Eva in $\mathcal{G}(\mathcal{Y}, v, \phi^{n+1}(\bot))$ from $(j_0, \phi^{n+1}(\bot))$.

Notice first that if $i \geq 1$, then the structure of the games $\mathcal{G}(\mathcal{Y}, v, \phi^{i+1}(\bot))$ and $\mathcal{G}(\mathcal{Y}, v, \phi^{i}(\bot))$ are, at the beginning, the same from the positions of the form $(j, \phi^{i+1}(\bot))$ and $(j, \phi^{i}(\bot))$. In particular, a winning strategy for Eva from $(j, \phi^{i+1}(\bot))$ in $\mathcal{G}(\mathcal{Y}, v, \phi^{i+1}(\bot))$ can be simulated, at the beginning, within the game $\mathcal{G}(\mathcal{Y}, v, \phi^{i}(\bot))$ from position $(j, \phi^{i}(\bot))$.

Therefore Eva plays as follows: she plays from $(j_0, \phi^{n+1}(\bot))$ as if she was playing in $(j_0, \phi^{n+2}(\bot))$, according to the same winning strategy. If, at some point,

- she hits a position $(j_i, \phi^{n-i+1}(\bot))$, where she is playing the winning strategy from position $(j_i, \phi^{n-i+2}(\bot))$ in the game $\mathcal{G}(\mathcal{Y}, v, \phi^{n+2}(\bot))$, and
- the position $(j_i, \phi^{n-i+1}(\bot))$ is also part of the winning strategy in the game $\mathcal{G}(\mathcal{Y}, v, \phi^{n+2}(\bot))$,

then she jumps: she continue simulating the winning strategy from $(j_i, \phi^{n-i+1}(\bot))$ in the game $\mathcal{G}(\mathcal{Y}, v, \phi^{n+2}(\bot))$.

Evidently, this strategy is losing for Eva if she cannot jump. That is, Adam can force a play that visits positions $(j_i, \phi^{n-i+1}(\bot))$, $i = 0, \ldots, n+1$, such that all the j_i are distinct. As the $j_i \neq j_{i+1}$, this means that Eva, in the play that has lead from $(j_i, \phi^{n-i+1}(\bot))$ to $(j_{i+1}, \phi^{n-(i+1)+1}(\bot))$, has went through some choice of the form $k' \in C \in M(k)$ with $k \neq k'$, i.e. kDk'. Thus, we can sketch the play forced by Adam as follows:

$$\begin{aligned}(j_0, \phi^{n+1}(\bot)) &\xrightarrow{D^+} (j_1, \phi^{n}(\bot)) \xrightarrow{D^+} \ldots \\ &\ldots \xrightarrow{D^+} (j_n, \phi(\bot)) \xrightarrow{D^+} (j_{n+1}, \bot)\end{aligned}$$

that is, we can witness the existence of a sequence the join-dependency relation of length at least $n+1$. This contradicts the fact that $L \in \mathcal{L}_n$. $\square$

It follows that $\mathcal{L}_n$ has a trivial least fixed-point theory. Namely, define μ-terms according to the following grammar:

$$t = x \mid \bot \mid t \vee t \mid \top \mid t \wedge t \mid \mu_z.t\,,$$

interpret these terms on finite lattice as usual with $\mu_z.t$ denoting the least fixed-point of the order preserving function – of the variable z – denoted by t. Then, $\mathcal{L}_n$ makes correct the following translation:

$$\begin{aligned}
\lceil x \rceil &= x\,, & & \\
\lceil \top \rceil &= \top\,, & \lceil t \wedge s \rceil &= \lceil t \rceil \wedge \lceil s \rceil\,, \\
\lceil \bot \rceil &= \bot\,, & \lceil t \vee s \rceil &= \lceil t \rceil \vee \lceil s \rceil\,, \\
\lceil \mu_x.\phi \rceil &= \lceil \phi \rceil^{n+1}(\bot)\,, & &
\end{aligned}$$

so that each fixed-point term is equivalent on $\mathcal{L}_n$ to a term with no fixed-points.

For each $n \geq 0$, let

$$\sigma(n) = \min\{\, i \geq 0 \mid \mathcal{L}_n \models \phi^{i+1}(\bot) = \phi^{i}(\bot)\,\}\,,$$

where we write $\mathcal{L}_n \models \phi^{i+1}(\bot) = \phi^{i}(\bot)$ to mean that the equation $\phi^{i+1}(\bot) = \phi^{i}(\bot)$ holds on every lattice in $\mathcal{L}_n$. Proposition 7.4 shows that $\sigma(n) \leq n+1$. The following Proposition exhibits a lower bound for σ:

$$\frac{n-2}{3} \leq \sigma(n)\,.$$

Proposition 7.5. *Let*

$$\phi(z) = a \wedge (b \vee (c \wedge (a \vee (b \wedge (c \vee z)))))\,.$$

For each $n \geq 0$ there exists an atomistic lattice $L \in \mathcal{L}_{3n+2}$, a valuation v, and an atom a of L such that $a \leq \phi^{n+1}(\bot)$ but $a \not\leq \phi^{n}(\bot)$.

Proof. Let us define a sequence $A_0, \ldots, A_{3n}$ of atomisitic presentations.

We let $A_0 = \langle \{\, v_0, w_0, v_{-1}, w_{-1}, v_{-2}\,\}, M \rangle$ where

$$\begin{aligned}
M(z) &= \{\, \{\, z \,\}\,\}\,, \quad z \in \{\, w_0, w_{-1}, v_{-2}\,\}\,, \\
M(v_{-1}) &= \{\, \{\, v_{-1}\,\}, \{\, w_{-1}, v_{-2}\,\}\,\}\,, \\
M(v_0) &= \{\, \{\, v_0\,\}, \{\, w_0, v_{-1}\,\}, \{\, w_0, v_{-1}, v_{-2}\,\}\,\}\,.
\end{aligned}$$

To define the presentation A_{n+1}, given A_n, let $A(w)$ denote the presentation of the powerset of $\{\, w \,\}$: $A(w) = \langle \{\, w \,\}, M \rangle$, with $M(w) = \{\, \{\, w \,\}\,\}$. Let us suppose $A_n = \langle V_n, M \rangle$ with $V_n = \{\, v_{-2}, v_{-1}, v_0, \ldots, v_n \,\} \cup \{\, w_{-1}, w_0, w_1, \ldots, w_n \,\}$. Let $\mathcal{F} = \{\, w_{n+1}, v_n \,\}$ and define

$$A_{n+1} = A(w_{n+1}) \oplus_{v_{n+1}, \mathcal{F}} A_n\,.$$

Clearly, $\mathfrak{L}(A, M) \in \mathcal{L}_{3n+2}$.

Next, for each $n \geq 0$, let $v : \{ a, b, c \} \to P(V_{3n})$ be the function defined as follows:

$$
\begin{aligned}
v(a) &= \{ v_i \mid n \equiv 0 \mod 3 \} \cup \{ w_i \mid i \equiv -1 \mod 3 \} \\
v(b) &= \{ v_i \mid i \equiv -2 \mod 3 \} \cup \{ w_i \mid i \equiv 0 \mod 3 \} \\
v(c) &= \{ v_i \mid i \equiv -1 \mod 3 \} \cup \{ w_i \mid i \equiv -2 \mod 3 \} \cup \{ \} .
\end{aligned}
$$

Claim 7.6: For each $z \in \{ a, b, c \}$, $v(z)$ is closed.

Let $C \subseteq v(z)$ with $C \in M(y)$. If C is not a singleton then for some $k = -1, \ldots, 3n$, $\{ w_k, v_{k-1} \} \subseteq C \subseteq v(z)$. It is easily seen that this is not the case, as by construction, if $w_k \in v(z)$ then $v_{k-1} \notin v(z)$. We deduce that C is a singleton, whence it is the singleton $\{ y \}$; form $\{ y \} \subseteq v(z)$ we deduce $y \in v(z)$.

Claim 7.7: For each $n \geq 0$, $v_{3n} \models_v \phi^{n+1}(\bot)$ and $v_{3n} \not\models \phi^n(\bot)$.

The proof is by induction on $n \geq 0$. Of course, $v_0 \not\models \phi^0(\bot) = \bot$. We leave the reader to verify that $v_0 \models \phi(\bot)$.

Next we suppose that $v_{3n} \models \phi^{n+1}(\bot)$ and $v_{3n} \not\models \phi^n(\bot)$. Again, we leave the reader to verify that $v_{3n+3} \models \phi^{n+2}(\bot)$. We verify next that $v_{3n+3} \not\models \phi^{n+1}(\bot)$. Suppose on the contrary that $v_{3n+3} \models \phi^{n+1}(\bot)$. Then there exists $C \in M(v_{3n+3})$ such that, if $y \in C$, then $y \models b$ or $y \models c$. As $\{ w_{3n+3}, v_{3n+2} \}$ is the only cover if v_{3n+3} which is colored just by b, c, we have $C = \{ w_{3n+3}, v_{3n+2} \}$, whence $v_{3n+2} \models_v c \wedge (a \vee (b \wedge (c \vee \phi^n(\bot))))$. Up to symmetry, the same argument shows that $v_{3n+1} \models_v b \wedge (c \vee \phi^n(\bot))$ and $v_{3n} \models_v \phi^n(\bot)$. Thus we obtain a contradiction, and deduce that $v_{3n+3} \not\models \phi^{n+1}(\bot)$. $\square$

8 Towards a correspondence theory

The goal of this section is twofold. On one side, we hint that many first order properties of an OD-graph correspond to equational properties of their lattices. We are far from understanding the general mechanism of this correspondence, but we believe exploring this direction is a research objective worth to be pursued. Our second goal is to generalize Nation's and Semenova's result stating that finite lattices, whose sequences of the relation D have length at most n, form a pseudovariety.

8.1 T-shapes

In the following T shall denote a fixed finite tree rooted at t_0.

Definition 8.1. A *T-shape* in $\langle J(L), \leq, \mathcal{M} \rangle$ is a function $f : T \to J(L)$ such that f is locally injective and sends sons to a partial cover. That is, for each $t \in T$,

1. $f : \{ t' \mid t \prec t' \} \to \{ f(t') \mid t \prec t' \}$ is a bijection,

2. there exists $C_t \in \mathcal{M}(f(t))$, $C_t \neq \{ j \}$, such that $\{ f(t') \mid t \prec t' \} \subseteq C_t$.

We shall say that there is a T-shape at $j \in J(L)$ if for some T-shape f we have $f(t_0) = j$.

Remark 8.2. Notice that we can express existence of a T-shape within the first order theory of the two-sorted structure $\langle J(L), \bigcup_{j \in J(L)} \mathcal{M}(j), \in, \triangleleft \rangle$, where $j \triangleleft C$ means that $C \in \mathcal{M}(j)$.

For each $t \in T$, let x_t, y_t be distinct variables. We define

$$U_t = \begin{cases} x_t \,, & \text{if } t \text{ is a leaf}\,, \\ x_t \wedge (y_t \vee \bigvee_{t \prec t'} U_{t'})\,, & \text{if } t \text{ is an internal node}\,. \end{cases}$$

We shall use the notation U_T for U_{t_0} where t_0 is the root of T.

A T-shape gives rise to an interpretation (we call it v_f) of the variables in U_T:

$$v_f(x_t) = f(t)\,, \qquad\qquad v_f(y_t) = \bigvee (\, C_t \setminus \{ f(t') \mid t \prec t' \} \,)\,.$$

Lemma 8.3. *The intepretation v_f satisfies the equation*

$$x_t = U_t\,,$$

for all $t \in T$.

Proof. That's clear if t is a leaf. Let us suppose that the equality holds for all t' such that $t \prec t'$. Let us prove it holds for t as well.

$$U_t = f(t) \wedge (v_f(y_t) \vee \bigvee_{t \prec t'} f(t')) = f(t) \wedge \bigvee C_t = f(t) = x_t\,,$$

since $C_t \in \mathcal{M}(f(t))$. □

Next, we define terms W_t as follows:

$$W_t = \bigvee_{t \prec t'} (x_t \wedge U_{t'}) \vee \bigvee_{t \prec t'} (x_t \wedge (y_t \vee W_{t'} \vee \bigvee_{\substack{t \prec \tilde{t} \\ t' \neq \tilde{t}}} U_{\tilde{t}}))\,.$$

As before, W_T shall denote the term W_{t_0}.

Lemma 8.4. *If f is a T-shape, then $f(t) \not\leq W_t(v_f)$, for each $t \in T$.*

Proof. The proof is by induction on the structure of the tree. This is clear if t is a leaf, since $f(t)$ is join-irreducible – hence distinct from $\perp$ – and $W_t = \perp$.

Suppose next that t is not a leaf. If $f(t) \leq W_t(v_f)$ then for some cover $C \in \mathcal{M}(f(t))$ such that, if $c \in C$ then $c \leq f(t)$ and $c \leq f(t')$ (since $U_{t'} = f(t')$)

or $c \leq \bigvee(C_t \setminus f(t'))$. However, the fact that $C \ll \{ f(t) \}$ implies $C = \{ f(t) \}$. We deduce therefore that for some t' with $t \prec t'$, either $f(t) \leq f(t')$ or

$$f(t) \leq \bigvee(C_t \setminus f(t')) \cup \{ W_{t'}(v_f) \} .$$

The first case cannot be, as $f(t') \in C_t$ which is a nontrivial cover of $f(t)$.

The second case cannot be, for the following reason. If this was the case, then – by minimality of the cover C_t – we would have

$$f(t') = U_{t'}(v_f) = W_{t'}(v_f) ,$$

contradicting the induction hypothesis.

We derive therefore that $f(t) \not\leq W_t(v_f)$. □

Proposition 8.5. *$j \in J(L)$ satisfies the equation*

$$U_T \leq W_T$$

if and only if there is no T-shape f with $f(t_0) = j$.

Proof. The previous Lemmas show that if j satisfies the equation $U_T \leq W_T$, then there is no T-shape f at j. Otherwise, for v_f we have $j = v_f(t_0) = U_{t_0}(v_f)$, and $j \not\leq W_{t_0}(v_f)$.

For the converse, we prove by induction on the structure of the tree that if there is no t-shape at j, then $U_t \leq W_t$ – for an arbitrary valuation v.

The statement trivially holds if t is a leaf, since the condition *there is no t-shape at j* is always false.

Let us suppose that t has a nonempty set of sons and that, for these sons, the statement holds. We suppose that there is no t-shape at j and that $j \leq U_t$, we shall show that $j \leq W_t$.

From $j \leq U_t$, we deduce that $j \leq x_t$ and that there exists $C \in \mathcal{M}(j)$ such that $C \ll \{ y_t \} \cup \{ U_{t'} \mid t \prec t' \}$. If $C = \{ j \}$, then $j \leq y_t$ or $j \leq U_{t'}$ for some t' with $t \prec t'$, and in both cases we derive $j \leq W_t$:

$$j \leq x_t \wedge y_t \leq W_t , \qquad\qquad j \leq x_t \wedge U_{t'} \leq W_t .$$

Otherwise C is a nontrivial cover and we can partition it as

$$C = P_t \uplus \biguplus \{ P_{t'} \mid t \prec t' \}$$

so that, if $c \in P_t$, then $c \leq y_t$, and if $c \in P_{t'}$, then $c \leq U_{t'}$. Notice that some of the elements of the partition, i.e. P_t or some $P_{t'}$, might be the emptyset.

Next, if for each t' such that $t \prec t'$ there is $c_{t'} \in P_{t'}$ with a t'-shape from c, then we can easily construct a t-shape from j – notice that we need here the $P_{t'}$ to be disjoint. As this is not the case, then there is at least one t' such that $t \prec t'$ and, moreover, if $c \in P_{t'}$, then there is no t'-shape from c. Then $c \in P_{t'}$ implies $c \leq U_{t'}$ and, by the inductive hypothesis, $c \leq W_{t'}$.

Therefore, we have $C \ll \{ y_t, W_{t'} \} \cup \{ U_{\tilde{t}} \mid t \prec \tilde{t}, \tilde{t} \neq t' \}$, thus

$$j \leq x_t \wedge (y_t \vee W_{t'} \vee \bigvee_{\substack{t \prec \tilde{t} \\ t' \neq \tilde{t}}} U_{\tilde{t}}) \leq W_t \,.$$

$\square$

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